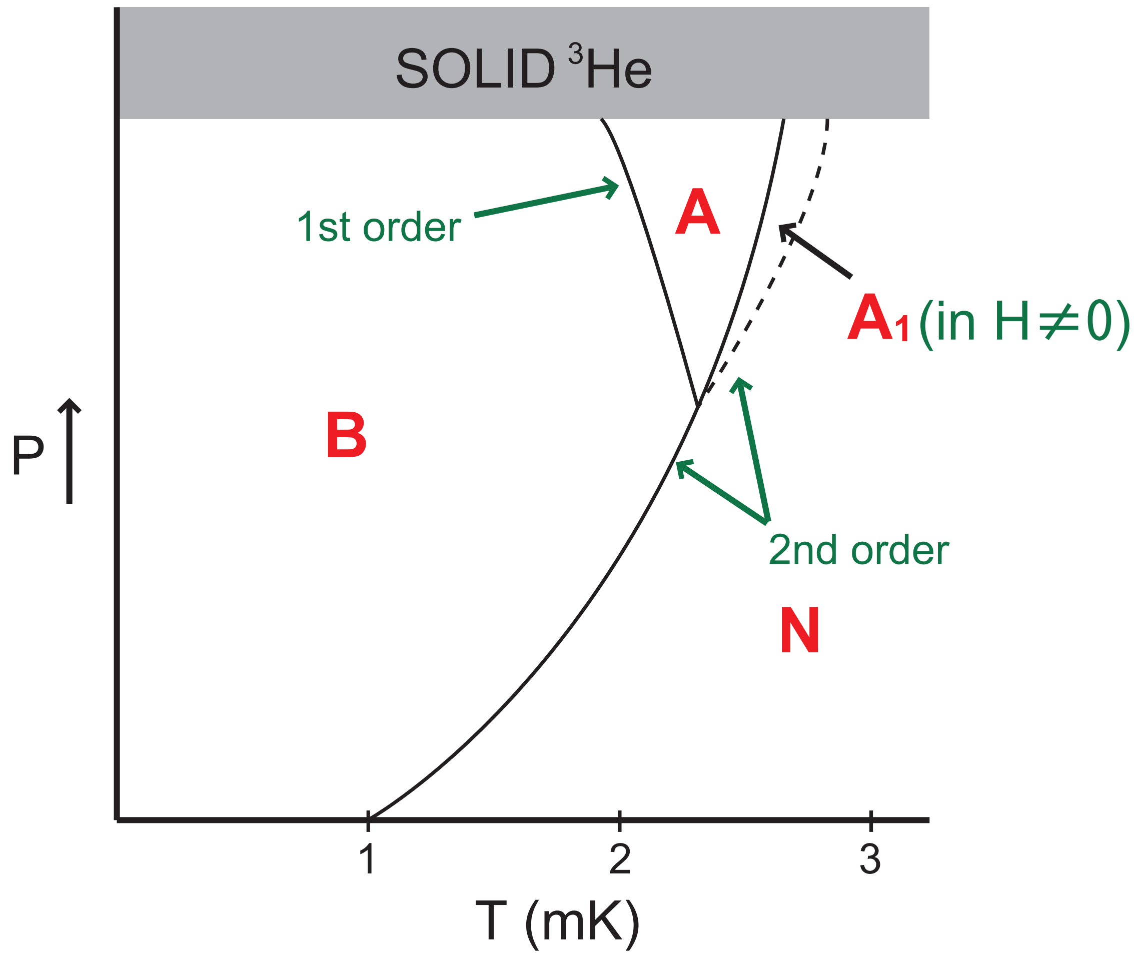
Lecture3 Superfluid ^{3}He cont.

Lecture Web page:
http://www.phys.s.u-tokyo.ac.jp/curriculum/H23shuchu-kobetsu_015.htm
room: 434

1. The experimental phases of liquid ^{3}He



Nature of Order Parameter of different phases : (stability, susceptibility, NMR,...)

A phase :

$$\boldsymbol{d}(\hat{\boldsymbol{k}}) = \hat{\boldsymbol{d}} \cdot \text{const.} \cdot (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\ell}}) \leftarrow \text{ABMstate} \tag{1}$$

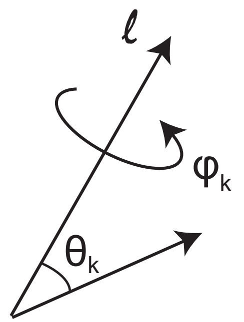
$\hat{\boldsymbol{d}}$: characterize spin vector $\hat{\boldsymbol{\ell}}$: characterize orbited vector

i.e. : $F(\boldsymbol{r}, \sigma) = F(\boldsymbol{r})\chi(\sigma)$

$\chi(\sigma) =$ state $S = 1,\ S_z = 0$ along $\hat{\boldsymbol{d}}$

$F(\boldsymbol{r}) =$ state with (apparent) angular momentum $\hbar$ along $\hat{\boldsymbol{\ell}}$

$\Delta_k = \Delta_0 \sin\theta_k \exp(i\phi_k) \Rightarrow$ nodes at $\pm\boldsymbol{\ell}$ $\Rightarrow$ many quasi-particle's at low T. $\boldsymbol{d}$ independent of $\hat{\boldsymbol{k}}$ $\Rightarrow$ equal spin pairing state, χ not reduced



φ_k , θ_k : ∠' s relative to ℓ

Alternative representation of ABM state :

$$\Psi_N = \text{const.}\left(\sum_{\boldsymbol{k}} c_k(\alpha^\dagger_{\boldsymbol{k}\uparrow}\alpha^\dagger_{-\boldsymbol{k}\uparrow} + \exp(i\chi)\alpha^\dagger_{\boldsymbol{k}\downarrow}\alpha^\dagger_{-\boldsymbol{k}\downarrow})\right)^{N/2}|\text{vac}\rangle \tag{2}$$

$c_k = f(|k|)\sin\theta_k\exp(i\phi_k)$ χ : half angle of d in xy plane

A_1 phase :

$$\Psi_N = \text{const.}\left(\sum_{\boldsymbol{k}} c_k\alpha^\dagger_{\boldsymbol{k}\uparrow}\alpha^\dagger_{-\boldsymbol{k}\uparrow}\right)^{N/4} \times (\downarrow \text{Fermi Surface}) \tag{3}$$

note : not automatically spin-polarized!

B phase : naive identification :

$\boldsymbol{d}(\hat{\boldsymbol{k}}) = \text{const.} \times \hat{\boldsymbol{k}} \leftarrow$ BW state
i.e. spin state is $S = 1, S_z = 0$ along $\boldsymbol{k}$ $(\hat{L}_z = 0)$
i.e. $\boldsymbol{S}$ always directed oppositely to $\boldsymbol{L}$:
$\boldsymbol{L} + \boldsymbol{S} \equiv \boldsymbol{J} = 0$, i.e.

$$^3P_0 \text{ state} \leftarrow \text{BW state}$$

by Wigner-Eckart theorem, all properties isotropic, in particular,

$$|\Delta(\boldsymbol{n})| \propto |\boldsymbol{d}(\boldsymbol{n})| = \text{const.} \tag{4}$$

$\Rightarrow$ no. of quasi-particles exponentially small ($\propto \exp(-\Delta/T)$) at low temperature.
Spin susceptibility χ : for any direction of field, some $S_z = 0$ pairs
$\Rightarrow$ χ reduced from N-state value.
(so, magnetic field advantages A phase over B phase)

Why the A phase?

Recall (1.2) : for pairing with given l, free energy is minimized by the choice which minimizes the anisotropy of $|\boldsymbol{d}(\hat{\boldsymbol{n}})|^2$ over the Fermi surface, or specifically (for unitary states)

$$\text{minimize } \overline{|\boldsymbol{d}|^4}/(\overline{|\boldsymbol{d}|^2})^2 \equiv K.$$

Now :

$$\text{for ABM, } |\boldsymbol{d}(\hat{\boldsymbol{n}})|^2 \propto \sin^2\theta \Rightarrow K = \frac{8/15}{(2/3)^2} = 6/5$$
$$\text{for BW, } |\boldsymbol{d}(\hat{\boldsymbol{n}})|^2 \propto \text{const.} \Rightarrow K = 1$$

$\Rightarrow$ in generalized BCS theory, F_{BW} always $< F_{\text{ABM}}$!

Possible approaches :
(a)generalized GL(Ginzburg-Landau) approach :
since $\boldsymbol{d}(\hat{\boldsymbol{n}})$ is a vector in spin space, and orbital dependence is assumed to be p-wave, can always write

$$d_\alpha(\hat{\boldsymbol{n}}) \equiv \sum_i d_{i\alpha}\hat{\boldsymbol{n}}_i \tag{5}$$

$\Rightarrow$any p-wave triplet states parametrized by 9 complex numbers $d_{i\alpha}$.

Now assume generalized GL expansion :

$$F(T : \{d_{i\alpha}\}) = (\text{const.}+)(T_c - T) \cdot \text{o}(|d|^2) + \beta(T) \cdot \text{o}(|d|^4) + ... \tag{6}$$

From invariance under spin + rotation separately(plus gauge invariance) :

$$\text{o}(|d|^2) \propto \sum_{i\alpha} |d_{i\alpha}|^2 \left(\equiv \int \frac{d\Omega}{4\pi} |\boldsymbol{d}(\hat{\boldsymbol{n}})|^2, \text{as in BCS} \right) \tag{7}$$

but for the $\text{o}(|d|^4)$ term 5 independent invariants, e.g.

$$I_1 = (\sum_{i\alpha} |d_{i\alpha}^2|)^2 \leftarrow \left(\text{note differennt from } (\sum_{i\alpha} |d_{i\alpha}|^2)^2! \right) \tag{8}$$

$$I_2 = \sum_{\alpha\beta ij} d_{\alpha i}^* d_{\alpha j} d_{\beta i}^* d_{\beta j} \quad \text{etc} \tag{9}$$

and in general

$$\text{o}(|d|^4) = \sum_{s=1}^{5} \beta_s I_s \tag{10}$$

Generalized GL approaches (cont.)(for details, see e.g. AJL RMP '75, section IX.B)

Recap : for given normalization of $\boldsymbol{d}(\boldsymbol{n})$ ($e.g. \sum_{i\alpha} |d_{i\alpha}|^2 \equiv \overline{|d|^2} = 1$)
must minimize expression

$$K \equiv \sum_{s=1}^{5} \beta_s I_s \quad (e.g. I_2 = \sum_{\alpha\beta ij} d^*_{\alpha i} d_{\alpha j} d^*_{\beta i} d_{\beta j}) \tag{11}$$

β_s : unknown coefficient

I_s : 4^{th} - order invariant characteristic of particular state(ABM, BW)

Problem : find all possible states (= forms of $d_{i\alpha}$) which can be free energy minima (i.e. minima of K) for some choice of the β_s.
Unrestricted problem : v. messy, near-insoluble.
So : restrict to unitary states. (for every $\hat{\boldsymbol{n}}$, $S_z = 0$ in some set of spin axis.)
Then, only 4 states can be extrema of free energy :
(a)BW ("isotropic"), $d_{\alpha i} = \frac{1}{\sqrt{3}} \delta_{\alpha i}$
(b)2D ("planar"), $d_{\alpha i} = \frac{1}{\sqrt{2}} \delta_{\alpha i}(1 - \delta_{iz})$
(c)ABM ("axial"), $d_{yx} = -i d_{zx} = \frac{1}{\sqrt{2}}$, all other $d_{\alpha i} = 0$
(d)1D ("polar"), $d_{zz} = 1$, all other $d_{\alpha i} = 0$

Theorem : (remarkable!)

2D (planar) state is never absolute minimum of free energy, for any choice of $\beta's$.
For BCS values of $\beta's$, BW state most stable (of course). Polar state competitive with BW only if non-BCS contributions to $\beta's$ comparable to BCS ones. But ABM state competitive with BW for relatively small non-BCS contributions.
⇒ in general, BW/ABM most likely.

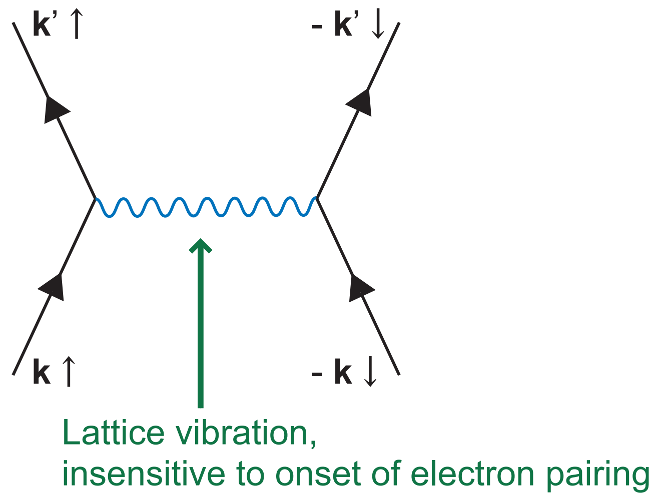
Important note : in above analysis, both α's and i's always occur in pairs.
⇒ $I_s's$ invariant under spin and orbital rotation separately. So BW, ABM, etc. represent classes of states transforming into one another under these rotations.

Possible approaches (cont.)

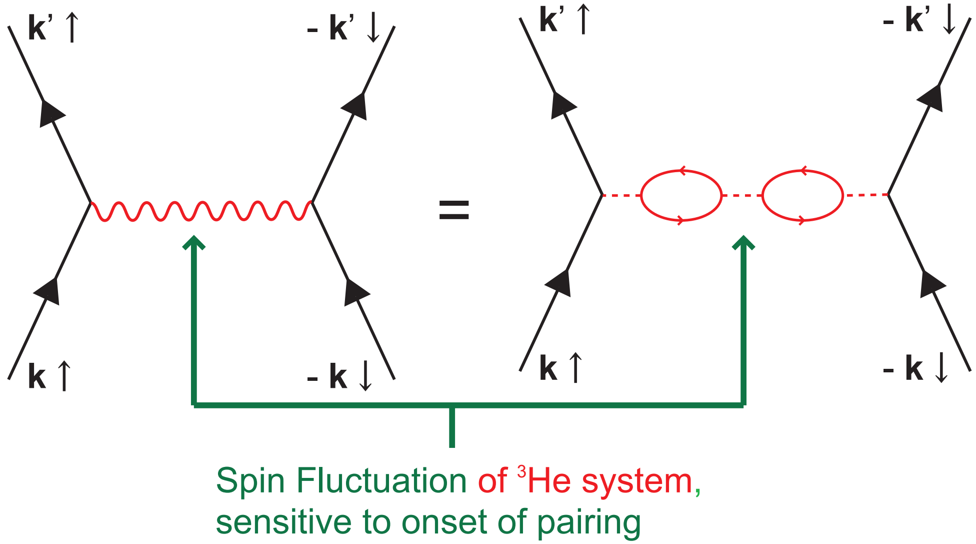
(b)Spin fluctuation feedback (Anderson+Brinkman, PRL **30**, 1108 (1973))

The physical idea :

Superconductor:



Liquid ^{3}He:



Technically : (cf. 1.2)
Spin Fluctuation induced interaction is

$$V_{\text{eff}}(\boldsymbol{q}\omega) \approx -(F_0^a)^2 \chi_{\text{sp}}(\boldsymbol{q}\omega)\boldsymbol{\sigma} \cdot \boldsymbol{\sigma'} \tag{12}$$

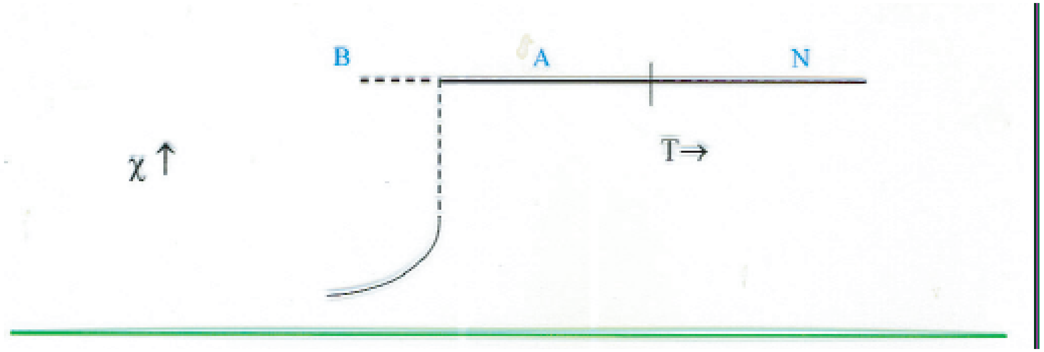
and $\chi_{\text{sp}}(\boldsymbol{q}\omega)$ is modified by pairing.
Crude ansatz : $\delta\chi_{\text{sp}}(\boldsymbol{q}\omega) \propto \delta\chi$. $\leftarrow$ static spin susceptibility.
In BW state, $\chi < \chi_n \;\Rightarrow\; V_{\text{eff}}$ reduced $\Rightarrow$ disfavored.
ABM state is more subtle : for fixed $\boldsymbol{d}$, $\delta\chi$ is actually anisotropic,

$$\delta\chi_{ij} \approx f(T) d_i d_j \tag{13}$$

$$\Delta\langle V\rangle \approx -\delta\chi_{ij}\langle\sigma_i\sigma_j\rangle \approx +f(T)d_i d_j\langle\sigma_i\sigma_j\rangle \approx \langle(\boldsymbol{\sigma}\cdot\boldsymbol{d})^2\rangle \tag{14}$$

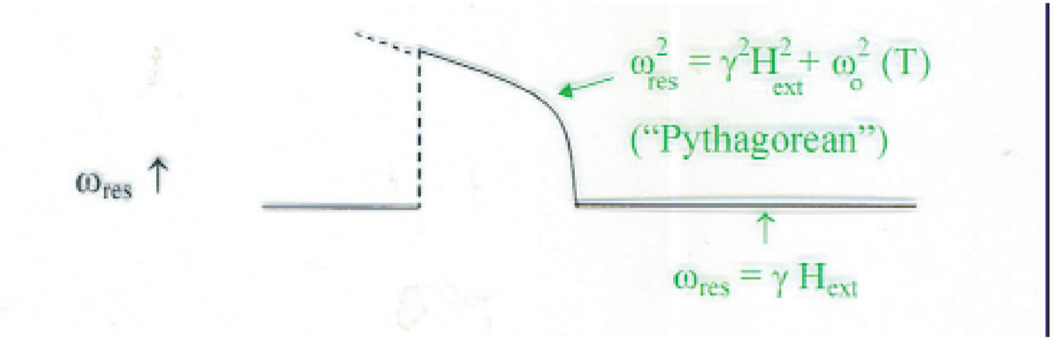
but, in paired state $(\boldsymbol{\sigma}\cdot\boldsymbol{d})^2$ is negative!
$\Rightarrow$Spin Fluctuation attraction increased in ABM state over N-state value.

NMR in the new phase :



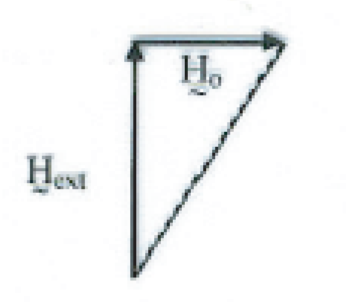
Not necessarily mysterious : e.g. A phase could be an ESP(Equal Spin Pairing) state (only ↑↑, ↓↓pairs ⇒ no reduction in χ), B could be singlet or BW (some ↑↓pairs, so χ reduced). But why is ESP ever stable?

But : what about the resonance frequency?



$$\omega_0^2(T) \approx A(1 - T/T_A), \ \frac{A}{(2\pi)^2} \approx 5 \times 10^{10}\mathrm{Hz}^2 \tag{15}$$

Need $H_0(\equiv \omega_0(T)/\gamma) \sim 30G$. But, only spin-non-conserving force in problem is nuclear dipole-dipole interaction, and max. associated field is < 1G!



Is this the first indication of radical breakdown of quantum mechanics?

What can be inferred from sum rules?

IF a single sharp resonance is observed (as in expection) then :

$$\omega_{\rm res}^2 = \gamma^2 H_{\rm ext}^2 + \omega_0^2 \tag{16}$$

$$\omega_0^2 = \gamma^2 \chi^{-1} \frac{\partial^2 \langle H_D \rangle}{\partial \theta^2} \tag{17}$$

H_D : nuclear dipole energy, θ: angle of simultaneous rotation of all spins

But $\frac{\partial^2 \langle H_D \rangle}{\partial \theta^2} \sim \langle H_D \rangle$: So, experimental value of $\omega_0^2(T) \Rightarrow$

$$\langle H_D \rangle (T) \sim K \left(1 - \frac{T}{T_A} \right), \; K \sim 10^{-3} \text{ergs/cm}^3 \tag{18}$$

How can this be?

$$\uparrow (\text{"}\textit{bad}\text{"}) \uparrow \qquad \uparrow \tag{19}$$

$$(\text{"}\textit{good}\text{"}) \tag{20}$$

$$\uparrow \tag{21}$$

$$\Delta E \leq \frac{\mu_0 \mu_n^2}{r_0^3} \sim 10^{-7}\mathrm{K} \ll k_B T \tag{22}$$

So, prima facie, preference for good orientation over "bad" is <u>at most</u>

$$\sim \Delta E / k_B T \sim 10^{-4} \; (\text{actually}, \sim \Delta E / k_B T_F \sim 10^{-7}) \tag{23}$$

$\Rightarrow$ expectation value of dipole energy much too small!

Spontaneously Broken Spin-Orbit Symmetry

Ferromagnetic analogy :

<u>Ferromagnet</u>

$$\hat{H} = \hat{H}_0 + \hat{H}_z$$

$\hat{H}_0$: invariant under simultaneous rotation of all spins

$$\hat{H}_z = -\mu_B \mathscr{H} \sum_i S_{zi}$$

breaks spin-rotation symmetry

Paramagnetic phase ($T > T_c$) :
spins behave independently,
$k_B T$ competes with $\mu_B \mathscr{H}$
⇒polarization $\mu_B \mathscr{H} / k_B T \ll 1$
$\Rightarrow \langle H_z \rangle \sim N \left(\mu_B \mathscr{H}\right)^2 / k_B T$

<u>Liquid ^{3}He</u>

$$\hat{H} = \hat{H}_0 + \hat{H}_D$$

$\hat{H}_0$: invariant under <u>relative</u> rotation of spin + orbital coordinate systems.

$$\hat{H}_D = g_d \sum_{ij} \left(\frac{\sigma_i \cdot \sigma_j - 3 \sigma_i \cdot \hat{\boldsymbol{r}}_{ij} \sigma_j \cdot \hat{\boldsymbol{r}}_{ij}}{\left(r_{ij}^3 / r_0^3\right)} \right)$$

breaks <u>relative</u> spin-orbit rotation symmetry.

Normal phase ($T > T_A$) :
pairs of spins behave independently
⇒polarization $g_D / k_B T \ll 1$
$\Rightarrow \langle H_D \rangle \sim N g_D^2 / k_B T$

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Ferromagnet

Ferromagnetic phase $(T < T_c)$:

$\hat{H}_0$ forces all spins to lie parallel

$\Rightarrow k_B T$ competes with $N\ \mu_B \mathscr{H}$

$\Rightarrow \langle S_z \rangle \quad 1 \Rightarrow \langle H_z \rangle \ N \mu_B \mathscr{H}$

Liquid ^{3}He

Ordered phase $(T < T_A)$:

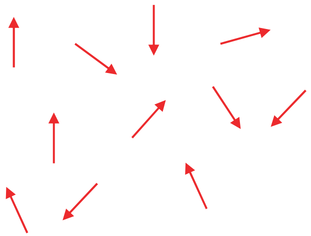
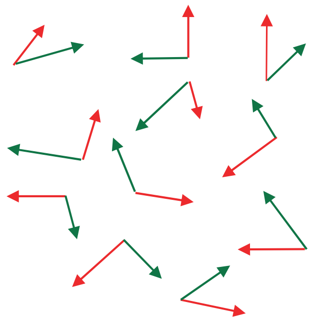
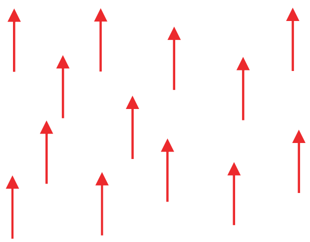
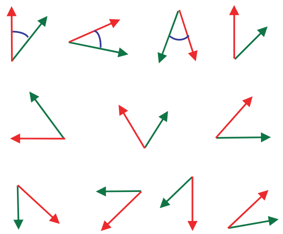
$\hat{H}_0$ forces all pairs to behave similarly

$\Rightarrow k_B T$ competes with $N g_D$

$\Rightarrow \langle H_D \rangle \quad N g_D$

10^{-3}ergs/cm^3!

SBOS : Ordering may be subtle

Ferromagnet	Liquid ^{3}He
Normal phase	Normal phase
	
Ordered phase	Ordered phase
	
	( = total spin of pair
	 = relative orbital ang. momentum)
$\langle \boldsymbol{S} \rangle \neq \mathbf{0}$	$\langle \boldsymbol{S} \rangle = 0$ $\langle \boldsymbol{L} \rangle = \mathbf{0}$
	but $\langle \boldsymbol{L} \times \boldsymbol{S} \rangle \neq \mathbf{0}$!!

Microscopic spin dynamics(schematic)

Basic variables :

(a)Total spin $\boldsymbol{S}$

(b)Orientation $\boldsymbol{\theta}$ of spin of Cooper pairs

$$\hat{H} = \hat{H}_0(\boldsymbol{S}) + \hat{H}_D(\boldsymbol{\theta}) \tag{24}$$

↑ hydrodynamic (Born-Oppenheimer) approximation

Semiclassical equations of motion :

$$\frac{d\boldsymbol{\theta}}{dt} = \frac{\partial\langle\hat{H}_0\rangle}{\partial\boldsymbol{S}} = \mathscr{H}_{ext} - \chi^{-1}\boldsymbol{S} \ (\mathscr{H}_{ext} : \text{zero in equilibrium.}) \tag{25}$$

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{S} \times \mathscr{H}_{ext} - \frac{\partial\langle\hat{H}_D\rangle}{\partial\boldsymbol{\theta}} \leftarrow \text{dipole torque} \tag{26}$$

⇒linear NMR behavior completely determined by eigenvalues of quantity

$$\Omega^2_{ij} \equiv \frac{\partial^2\langle H_D\rangle}{\partial\theta_i\partial\theta_j} \tag{27}$$

so, can "fingerprint" A and B phases by NMR!!

ABM : single resonance line

planer : split resonance

BW : original BW state is $\boldsymbol{L} = -\boldsymbol{S}$, *i.e.* $\boldsymbol{J} = 0$.
But dipole torque rotates $\boldsymbol{S}$ relative to $\boldsymbol{L}$ by $\angle \cos^{-1}(-1/4) = 104°$
around axis $\hat{\boldsymbol{\omega}}$ whose "best" choice is $\mathscr{H}_{ext}$.

Result : no shift in transverse resonance,
but finite-frequency longitudinal resonance!(also in ABM phase)

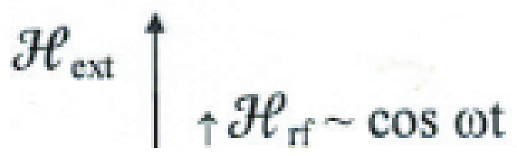


Illustration of NMR behavior : A-phase longitudinal resonance

Throughout the experiment, $\boldsymbol{d}$ lies in the xy-plane (but may rotate), i.e. schematically (ignore (fixed) orbital part of pair wave function)

$$\Psi_N(t) \approx \left(\exp\left(i\Delta\phi(t)/2\right)|\uparrow\uparrow\rangle + \exp\left(-i\Delta\phi(t)/2\right)|\downarrow\downarrow\rangle\right)^{N/2} \tag{28}$$

$$\Delta\phi(t) : \text{ note coherent superposition of } |\uparrow\uparrow\rangle \text{ and } |\downarrow\downarrow\rangle$$

Variable canonically conjugate to $\hat{\Delta\phi}$ is z-component of total (not Cooper-pair!) spin $\hat{S}_z$

$$[\hat{S}_z, \hat{\Delta\phi}] = 2i \tag{29}$$

Dipole interaction scatters $|\uparrow\uparrow\rangle \rightleftharpoons |\downarrow\downarrow\rangle$, so depends on relative phase $\Delta\phi$:

$$\hat{H}_D = -\frac{1}{4} g_D \cos \hat{\Delta\phi} \tag{30}$$

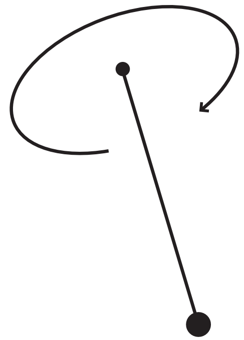
In adiabatic (BO) approx, polarization energy is

$$\hat{H}_0 = \hat{S}_z^2/2\chi - \hat{S}_z \mathscr{H}(t) \tag{31}$$

Eqns.(27-29) define problem of (driven) simple quantum pendulum.
Generally, in real-life experiments, in semiclassical limit (cf. below), so can treat $S_z(t)$ and $\Delta\phi(t)$ as classical variables. Then,

$$\frac{dS_z}{dt} = -\frac{\partial H}{\partial(\Delta\phi)} = -\frac{1}{2} g_D \sin\Delta\phi \tag{32}$$

$$\frac{d}{dt}(\Delta\phi) = \frac{\partial H}{\partial S_z} = 2\left(\frac{S_z}{\chi} - \mathscr{H}(t)\right) \tag{33}$$



small oscillations :
$\omega_0 = (g_D/\chi)^2$

rotation

"internal Josephson effect"

Digression : Spontaneous (?) Symmetry Breaking

Standard statement : global U(1) symmetry is spontaneously broken in superconducting (superfluid) state.
$\uparrow$: a priori experimentally untestable!

So consider breaking of relative symmetry, e.g. regarding relative phase of $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$.

We assume (cf. eqn. (26))

$$\Psi_N \sim \left(\exp(i\Delta\phi/2)\,|\uparrow\uparrow\rangle + \exp(-i\Delta\phi/2)\,|\downarrow\downarrow\rangle\right)^{N/2} \tag{A}$$

$\Delta\phi$: definite relative phase

Why not

$$\Psi_N \sim (|\uparrow\uparrow\rangle)^{N_\uparrow}(|\downarrow\downarrow\rangle)^{N_\downarrow} \quad ? \tag{B}$$

$N_\uparrow$, $N_\downarrow$: definite relative number ($S_z = N_\uparrow - N_\downarrow$)

Answer :

$$\hat{H} = \frac{\hat{S}_z^2}{2\chi} - \frac{1}{4} g_D \cos\hat{\Delta\phi} \tag{34}$$

$$\text{(a)} g_D \gg \chi^{-1} : \Delta\phi \text{ fixed, } S_z \text{ fluctuates} \quad \text{(case(A))}$$
$$\text{(b)} g_D \ll \chi^{-1} : S_z \text{ fixed, } \Delta\phi \text{ fluctuates} \quad \text{(case(B))}$$

But : $g_D \approx N$, $\chi^{-1} \approx N^{-1}$

$\Rightarrow$ in thermodynamic limit, always case (A)

To get case (B), need $N < 10^9$ (may be possible in small inclusions of liquid ^{3}He in solid ^{4}He)

Conclusion : No mystery about spontaneous symmetry breaking - just energetics!

Superfluid ^{3}He : Supercurrents, Textures, Defects...

In absence of "small" perturbations (inc. dipole forces) , A phase is characterized by a spin orientation vector $\boldsymbol{d}$ and on orbital orientation vector $\boldsymbol{\ell}$, which in uniform situation are arbitrary. Similarly, B phase is characterized by a rotation $\hat{\boldsymbol{R}}$ away from the $^3P_0$ (reference) state ; $\hat{\boldsymbol{R}}$ again is arbitrary .

Dipole force (dominant perturbation in bulk) tends, in A phase, to orient $\boldsymbol{d}$ $\parallel$ (or anti-$\parallel$) to $\boldsymbol{\ell}$, but common orientation. still arbitrary. In B phase fixes angle of rotation $\boldsymbol{R}$ the $\cos^{-1}(-1/4) = 104°$, but axis ($\boldsymbol{\omega}$) is still arbitrary.

External magnetic field $\boldsymbol{H}$ tends to orient $\boldsymbol{d} \perp$ to $\boldsymbol{H}$, and, in B phase, $\hat{\boldsymbol{\omega}} \parallel$ to $\boldsymbol{H}$. (very weak effect)

Spatial variation :

(a) If $\boldsymbol{d}$ and $\boldsymbol{\ell}$ (or $\hat{\boldsymbol{\omega}}$) fixed, only thing that can vary in overall phase ϕ : $\phi = \phi(\boldsymbol{R}) \Rightarrow$ ordinary superfluid (mass) flow.

$$\boldsymbol{v}_s(\boldsymbol{R}) = \frac{\hbar}{2m}\boldsymbol{\nabla}\phi(\boldsymbol{R}) \text{ (cf.superconductors, } ^4\text{He)} \tag{35}$$

only special point : in A phase superfluid density $\boldsymbol{\rho_s}$ is tensor with axis df. by $\boldsymbol{\ell}$ ($\rho_{s\perp} > \rho_{s\|}$ more quasi-particle's excited $\|$ to $\boldsymbol{\ell}$). Expect vortices as in ^{4}He.

(b) If ϕ and $\boldsymbol{\ell}$ fixed but $\boldsymbol{d}$ spatially varying (or ϕ fixed but $\hat{\boldsymbol{\omega}}$ varying) $\Rightarrow$ spin supercurrents.

(c) What if (in A phase) $\boldsymbol{d}$ is fixed but $\boldsymbol{\ell}$ (and ϕ?) vary in space?
Problem : since $F(\boldsymbol{r}) \sim f(r)\sin\theta_k \exp(i\phi_k)$, overall phase rotation is equivalent to rotation around axis of $\boldsymbol{\ell}$. But if $\boldsymbol{\ell}$ varies in space, such rotations are not holonomic
in particular, $\boldsymbol{\nabla} \times \boldsymbol{v_s} \neq 0$ in general.

Mermin - Ho rotation :

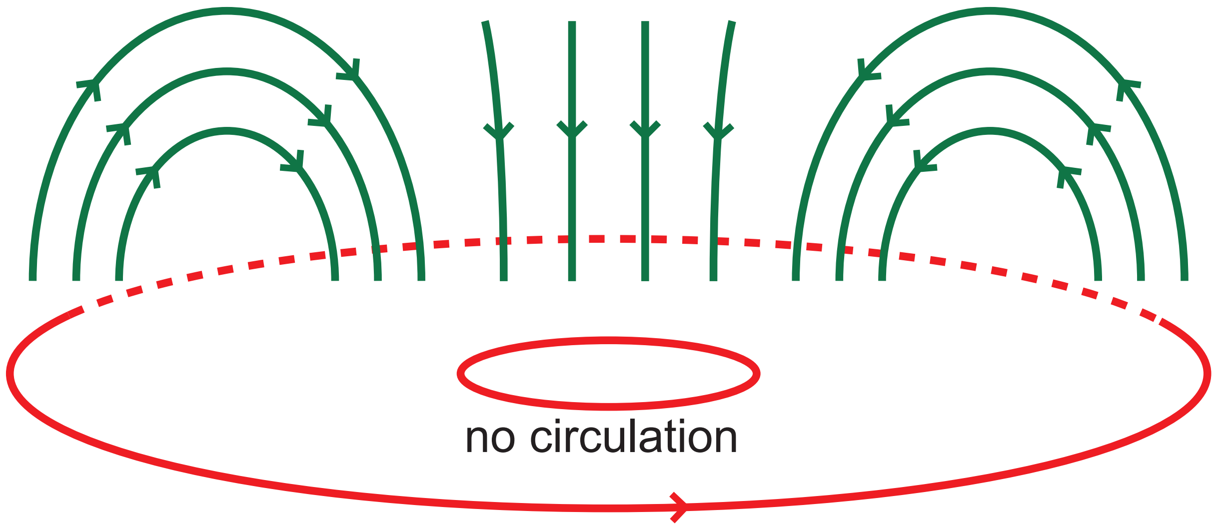
If we continue to define $\boldsymbol{v_s}$ in terms of infinitesimal rotations around $\boldsymbol{\ell}$, i.e. by

$$\boldsymbol{v_s} \equiv \frac{\hbar}{2m} \boldsymbol{\nabla} \phi \tag{36}$$

then

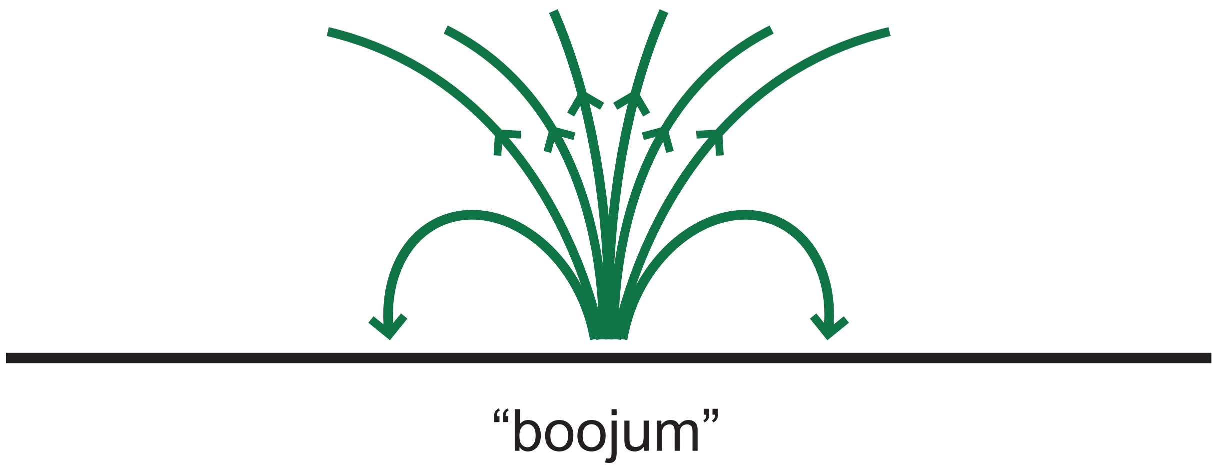
$$\boldsymbol{\nabla} \times \boldsymbol{v_s} = \frac{\hbar}{4m} \sum_{ijk} \epsilon_{ijk} \ell_i \boldsymbol{\nabla} \ell_j \times \boldsymbol{\nabla} \ell_k \tag{37}$$

$\Rightarrow$ possibility of "coreless" vortices



4π of circulation

(Cf : spin texture in 2D topological insulator.)



Energetics $\Rightarrow$ $\boldsymbol{\ell}$ must lie $\perp$ to walls :

For A phase in a simply connected geometry (e.g. cylinder) topological argument

$\Rightarrow$ must have at least 2 topological singularities (most likely boojums).

Also : all sorts of nonsingular textures (of $\boldsymbol{d}, \boldsymbol{\ell}, \boldsymbol{\omega}$..)