Ginzburg-Landau Theory (spin singlet pairing, for spatially uniform situation only)

Consider a general BCS-paired state (not necessarily the ground state) and define quantity

$$\Psi(\hat{\boldsymbol{n}}) \equiv \sum_{|\boldsymbol{k}|} F_{\boldsymbol{k}}, \quad F_{\boldsymbol{k}} \equiv \langle a^\dagger_{\boldsymbol{k}\uparrow} a^\dagger_{-\boldsymbol{k}\downarrow} \rangle = u_{\boldsymbol{k}} v_{\boldsymbol{k}}$$

The (pairing) potential energy $\langle V \rangle$ is given by the expression

$$\langle V \rangle = \sum_{\boldsymbol{k}\boldsymbol{k}'} V_{\boldsymbol{k}\boldsymbol{k}'} F_{\boldsymbol{k}} F^*_{\boldsymbol{k}'} = \int \frac{\mathrm{d}\Omega}{4\pi} \int \frac{\mathrm{d}\Omega'}{4\pi} V(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}') \Psi(\hat{\boldsymbol{n}}) \Psi^*(\hat{\boldsymbol{n}}')$$

We now confine ourselves to the case when $\Psi(\hat{\boldsymbol{n}})$ contains only spherical harmonics $Y_{\ell m}(\hat{\boldsymbol{n}})$ corresponding to the most negative harmonic V_ℓ of $V(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')$. Then $\langle V \rangle$ can be re-expressed in the form

$$\langle V \rangle = V_\ell \int \frac{\mathrm{d}\Omega}{4\pi} |\Psi(\hat{\boldsymbol{n}})|^2$$

What about kinetic energy $\langle K - \mu N \rangle = \sum_{\boldsymbol{k}\sigma} \varepsilon_{\boldsymbol{k}} \langle n_{\boldsymbol{k}\sigma} \rangle$, and the entropy $-TS$? We choose the $u_{\boldsymbol{k}}, v_{\boldsymbol{k}}$ to minimize these subject to getting the given value of $\Psi(\hat{\boldsymbol{n}})$. Then it is clear that $\langle K - \mu N \rangle - TS$ is a sum of contributions from different angles $\hat{\boldsymbol{n}}$ of the Fermi surface:

$$\langle K - \mu N \rangle - TS = \int \frac{\mathrm{d}\Omega}{4\pi} f\{\Psi(\hat{\boldsymbol{n}})\}$$

Moreover, from symmetry considerations,

$$f\{\Psi(\hat{\boldsymbol{n}})\} = f\{|\Psi(\hat{\boldsymbol{n}})|^2\} = \text{const.} + \alpha(T)|\Psi(\hat{\boldsymbol{n}})|^2 + \frac{1}{2}\beta(T)|\Psi(\hat{\boldsymbol{n}})|^4 + o(|\Psi(\hat{\boldsymbol{n}})|^6)$$

Hence the total free energy $F \equiv \langle K - \mu N + V \rangle - TS$ is

$$F = \text{const.} + (V_\ell + \alpha(T)) \int \frac{\mathrm{d}\Omega}{4\pi}|\Psi(\hat{\boldsymbol{n}})|^2 + \frac{1}{2}\beta(T) \int \frac{\mathrm{d}\Omega}{4\pi}|\Psi(\hat{\boldsymbol{n}})|^4 + o(|\Psi(\hat{\boldsymbol{n}})|^6)$$

We can work out the detailed form of $\alpha(T)$ and $\beta(T)$ if necessary, but will not need them.

The coefficient of $|\Psi(\hat{\boldsymbol{n}})|^2$ changes sign at the point where

$$V_\ell + \alpha(T) = 0$$

and this defines the critical temperature T_c. Close to this point, the coefficient is $\alpha'(T_c)(T_c - T)$, so

$$F = \text{const.} + \alpha'(T_c)(T_c - T) \int \frac{\mathrm{d}\Omega}{4\pi}|\Psi(\hat{\boldsymbol{n}})|^2 + \frac{1}{2}\beta(T) \int \frac{\mathrm{d}\Omega}{4\pi}|\Psi(\hat{\boldsymbol{n}})|^4 + o(|\Psi(\hat{\boldsymbol{n}})|^6)$$

$\beta(T) \approx \beta(T_c) \equiv \beta$. Note V_ℓ has fallen out of the problem! (in favor of T_c).

For s-wave pairing $\Psi(\hat{\boldsymbol{n}}) = \text{const.} = \Psi$, so we get the standard GL expression

$$F = \text{const.} + \alpha'(T_c)(T_c - T)|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + o(|\Psi|^6)$$

and minimizing with respect to Ψ gives $F = \text{const.} - (\alpha'(T_c)^2/2\beta)(T_c - T)^2$ in agreement with BCS.

For $\ell \neq 0$ pairing F depends on the form of $\Psi(\hat{\boldsymbol{n}})$, which is not unique. By minimizing with respect to the overall magnitude of Ψ, we find

$$F = \text{const.} - (\alpha'^2/2\beta)\left(\overline{|\Psi|^2}\right)^2 / \overline{|\Psi|^4}$$

where $\overline{g} \equiv \int \frac{\mathrm{d}\Omega}{4\pi} g(\hat{\boldsymbol{n}})$. Hence, the lowest free energy is obtained by minimizing $\overline{|\Psi|^4}$ subject to given $\overline{|\Psi|^2}$, i.e. by minimizing anisotropy of $|\Psi|^2$ over Fermi surface.

The spin triplet case:

Define

$$\boldsymbol{d}(\hat{\boldsymbol{n}}) \equiv \sum_{|\boldsymbol{k}|} \boldsymbol{d}_{\boldsymbol{k}}$$

then

$$\langle V \rangle = \int \frac{\mathrm{d}\Omega}{4\pi} \int \frac{\mathrm{d}\Omega'}{4\pi} V(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}') \boldsymbol{d}(\hat{\boldsymbol{n}}) \cdot \boldsymbol{d}^*(\hat{\boldsymbol{n}}') \to V_\ell \int \frac{\mathrm{d}\Omega}{4\pi} |\boldsymbol{d}(\hat{\boldsymbol{n}})|^2$$

and

$$\langle K - \mu N \rangle - TS = \int \frac{\mathrm{d}\Omega}{4\pi} f\{|\boldsymbol{d}(\hat{\boldsymbol{n}})|^2\}, \ \text{etc...}$$

So

$$F = \text{const.} + \alpha'(T_c)(T_c - T) \int \frac{\mathrm{d}\Omega}{4\pi} |\boldsymbol{d}(\hat{\boldsymbol{n}})|^2 + \frac{1}{2}\beta \int \frac{\mathrm{d}\Omega}{4\pi} |\boldsymbol{d}(\hat{\boldsymbol{n}})|^4 + o(|\boldsymbol{d}(\hat{\boldsymbol{n}})|^6)$$

(i.e. $|\Psi(\hat{\boldsymbol{n}})|^2 \to |\boldsymbol{d}(\hat{\boldsymbol{n}})|^2$)

$\Rightarrow$ minimize anisotropy of $|\boldsymbol{d}(\hat{\boldsymbol{n}})|^2$ over Fermi surface.