Lecture A3

Chern-Simons gauge theory

The Chern-Simons (CS) gauge theory in three dimensions is defined by the action,

$$\begin{aligned} S_{CS} &= \frac{k}{4\pi}\int \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right), \\ &= \frac{k}{8\pi}\int \epsilon^{\mu\nu\rho}\mathrm{tr}\left(A_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \frac{2}{3}A_\mu[A_\nu, A_\rho]\right). \end{aligned} \quad (1)$$

where tr is the trace over the fundamental representation of the gauge group G and k is a parameter of the theory (inverse of the coupling constant). If G is compact and simple, k has to be an integer in order for the action to be gauge invariant. (This has to do with the fact that $\pi_3(G) = \mathbf{Z}$ for any such group, where π_3 is the 3rd homotopy group – to be introduced in Lecture 10.)

The CS theory is an example of a topological field theory since writing down its action does not require a metric. If the quantization can be carried out without introducing a metric, observables of the theory would give topological invariants.

abelian CS theory

As a warm-up exercise, let us study the case when $G = U(1)$, *i.e.* the abelian CS theory. It has the action,

$$S_{U(1)} = \frac{k}{4\pi}\int A \wedge dA,$$

with A being the $U(1)$ connection. Note that $U(1)$ is not a simple group and k is not quantized in this case.

Such an action appears, for example, as the low energy effective theory to describe the topological order in fractional quantum Hall effect states. The equation of motion for the action simply says that $F = dA = 0$, which means that there is no local gauge-invariant observables. However, there are non-local observables. An important class of observables are the Wilson loops. Suppose that we are in $\mathbf{R}^3$. For each closed loop $\gamma \in \mathbf{R}^3$, we can consider the operator,

$$W(q;\gamma) = \exp\left(iq\oint_\gamma A\right).$$

Suppose we deform the loop infinitesimally. This can be done by attaching a small loop ϵ as $\gamma \to \gamma + \epsilon$. The Wilson loop changes as

$$W(q;\gamma+\epsilon) = \left(1 + iq\oint_\epsilon A\right)W(q;\gamma) = \left(1 + iq\int_D F\right)W(q;\gamma),$$

where D is a small disk in $\mathbf{R}^3$ such that $\partial D = \epsilon$ and we used the Stokes theorem. Since $F = 0$ by the equations of motion, the expectation value of $W(q;\gamma)$ is invariant under infinitesimal deformation of γ. Namely, it is a topological invariant of γ. Since the functional integral is Gaussian, one can evaluate the expectation value of products of these operators exactly. For example,

$$\frac{\langle W(q_1;\gamma_1)W(q_2;\gamma_2)\rangle}{\langle W(q_1;\gamma_1)\rangle\langle W(q_2;\gamma_2)\rangle} = \exp\left(-q_1q_2\oint_{\gamma_1} dx^\mu \oint_{\gamma_2} dy^\nu \langle A_\mu(x)A_\mu(y)\rangle\right) = \exp\left(\frac{2\pi i}{k}q_1q_2\Phi(\gamma_1,\gamma_2)\right),$$

where

$$\Phi(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} dx^\mu \oint_{\gamma_2} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}.$$

This is called the Gauss linkng number. It counts the number of links of the two loops (γ_1 and γ_2) by either $+1$ or -1 depending on the orientation of each link. This is one of the classical invariants of knots and links.

non-abelian CS theory

New features appear when the gauge group G is non-abelian. First of all, the action acquires the cubic term A^3, generating interactions. The defintion of the Wilson loop operator requires a care since A's do not commute with each other. Suppose the loop γ starts and end at a point x. Consider the vector bundle in a representation R of the structure group G and the connection A. Pick a vector v at the fiber over x. Now parallel transport v along the loop γ using the connection and come back to x. This gives another vector at the fiber over x. Thus, the parallen transport around γ defines a linear map on the fiber over x. The Wilson loop is defined as a trace of this linear map. More explicitly, we can also define the Wilson loop as

$$W(R;\gamma) = \mathrm{tr}_R P \exp\left(i \oint_\gamma A \right),$$

where tr_R indicates the trace over the representation space R, and the symbol P means the path-ordering.

In the simplest non-abelian case when $G = SU(2)$ and R is its fundamental representation, the expectation value of the Wilson loops gives the Jones polynomial invariants of knots and links. Below, we will see how one can compute such invariants.

In addition to invariants of knots and links, one can use the partition function (vacuum amplitude) of the CS theory to define a topological invariant of a 3-dimensional manifold M - the Reshetikhin-Turaev invariant.

canonical quantization

To compute amplitudes of the CS theory, it is often convenient to cut the manifold M into two parts across a two-dimensional surface. One reason for its usefulness is the existence of the Heegaard splitting. First define a handlebody as a 3-dimensional manifold with a boundary Σ such that cutting each of its handles across a disk produces the 3-sphere. Another way to think about a handle body is to embed a Riemann surface Σ in $\mathbf{R}^3$ and consider its interior plus the boundary Σ. Consider two such handlebodies with the same boundary Σ. Gluing them across the boundary produces a closed 3-dimensional manifold. It can be shown that any 3-dimensional differentiable manifold can be constructed in this way.

For example, consider a solid torus $T^2_{\alpha,\beta}$ with two homology cycles α, β and α is contractible. If we glue two copies of them together, we find $S^1 \times S^2$. On the other hand, if we glue $T_{\alpha,\beta}$ with $T_{\beta,\alpha}$ together, we get S^3.

As a special case, consider a three manifold of the form $\mathbf{R} \times \Sigma$ and perform the canonical quantization with $\mathbf{R}$ as time. This produces a Hilbert space $\mathcal{H}_\Sigma$ associated to Σ. In particular, if M_a ($a = 1, 2$) are handlebodies with $\partial M_a = (-1)^a \Sigma$ (the sign refers to the orientation), the CS functional integarls over M_a give particular state functions $\Psi_{a=1,2}$ in $\mathcal{H}_\Sigma$. The partition function for the closed manifold with the Heegaard splitting intwo $M_1 \cup M_2$ is then given by the inner product of the two wave functions (Ψ_1, Ψ_2).

Let us denote the connection in the $\mathbf{R}$ direction and in the Σ direction by A_0 and $A_{i=1.2}$, respectively. The CS action on $\mathbf{R} \times \Sigma$ can then be written as

$$S_{CS} = \frac{k}{4\pi} \int dt \int_\Sigma \mathrm{tr} \left(\epsilon^{ij} A_i \frac{\partial}{\partial t} A_j + A_0 F_{ij} \right),$$

where $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$, the curvature of the vector bundle over Σ. In this action, A_0 serves as the Lagrange multiplier to enforce the constraint $F_{ij} = 0$ and the gauge connection A_i is canonically conjugate to $\frac{k}{2\pi}\epsilon^{ij} A_j$. Namely, we are dealing with a system with constraints.

When we have a system with constraints, there are two ways to quantize it:

(1) Start with the canonical commutations relations, in this case,

$$[A_i^a(x), A_j^b] = \frac{2\pi i}{k} \epsilon_{ij} \delta^{ab} \delta^2(x-y),$$

where a, b are gauge group indices, and impose $F_{ij} = 0$ as conditions on physical wave functions.

(2) Impose the constraint first. In our case, the resulting space is the space of flat vector bundles over Σ. On this space, one can define a non-degenerate symplectic form, which one can derive by applying the symplectic reduction of the canonical symplectic form on A_i^a, and then quantize the constrainted phase space.

These two procedures should give the same answer.

When $G = SU(2)$, the dimensions of the Hilbert space is given by the Verlinde formula,

$$\mathrm{dim}\mathcal{H}_\Sigma = \left(\frac{k+2}{2} \right)^{g-1} \sum_{j=0}^{k} \left(\sin \frac{(j+1)\pi}{k+2} \right)^{2-2g},$$

where g is the genus (the number of handles) of Σ. There is a generalization of this formula for compact G.

holomorphic quantization

In the undergraduate quantum mechanics, one considers the space of square integrable functions of $q \in \mathbf{R}$. The position operator is the multiplication by q, and its conjugate momentum is the derivative operator $p = -i\partial_q$. We can exchange the role of q and p, consider the space of square integrable functions of p and regard q as the differential operator, $i\partial_p$. We can also consider their hybrid. Choose complex coordinate,

$$u = q + ip, \quad \bar{u} = q - ip.$$

Use holomorphic functions of u for wave functions, and regard $\bar{u}$ as the derivative operator ∂_u. This is called the holomorphic quantization.

This approach is useful here since we can make use of the complex structure $(z, \bar{z})$ on the Riemann surface Σ. The commutation relations between A_i^a's can be writte in the complex coordinates as,

$$[A_z^a(z,\bar{z}), A_{\bar{z}}^b(w,\bar{w})] = \frac{2\pi}{k} \delta^{ab} \delta(z-w)\delta(\bar{z}-\bar{w}).$$

We can then consider the Hilbert space as the space of holomorphic functionals $\Psi(A)$ of $A_{\bar{z}}$ and regard A_z as the functional derivative,

$$A_z^a = \frac{2\pi}{k}\frac{\delta}{\delta A_{\bar{z}}^a}.$$

The constraint $F_ij = 0$ can then be expressed as conditions on wave functions as,

$$\left(\partial_{\bar{z}}\frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}}\right]\right)\Psi(A) = \frac{k}{2\pi}\partial_z A_{\bar{z}}\Psi(A).$$

The Hilbert space consists of normalizable solutions to these equations.

WZW model

It turns out that the Hilbert space $\mathcal{H}_\Sigma$ for the canonical quantization of the CS theory on $\mathbf{R}\times\Sigma$ is naturally related to the two-dimensional conformal field theory called the WZW model on Σ. For simplicity, here we will discuss the case with $G = SU(2)$.

In Lecture A2, we discussed the Virasoro generators L_n derived from the energy-momentum tensor T_{zz}. The energy-momentum tensor T_{zz} is meromorphic and obeys the product operator expansion,

$$T_{zz}T_{ww} \sim \frac{c/2}{(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w\right)T_{ww} + 0(1).$$

The WZW model is a conformal field theory with holomorphic currents J_z^a ($a = 1, 2, 3$) in the adjoint representation of $SU(2)$. They obey the product operator expansion,

$$J_z^a\ J_w^b \sim \frac{k/2\delta^{ab}}{(z-w)^2} + \frac{\epsilon^{abc}}{z-w}J_w^c + 0(1).$$

By using

$$\partial_{\bar{z}}\frac{1}{z-w} = -\pi\delta^2(z-w),$$

which we derived earlier, we can also write this as,

$$\partial_{\bar{z}}J_z^a\ J_w^b = \frac{k\pi}{2}\delta^{ab}\partial_z\delta^2(z-w) - \pi\epsilon^{abc}\delta^2(z-w)J_w^c.$$

Consider couling these currents to an $SU(2)$ gauge field on Σ. For now, let us just turn on the anti-holomorphic component $A_{\bar{z}}$. Conside the partition function with $A_{\bar{z}}$,

$$Z(A) = \langle\exp(\pi\int_\Sigma A_{\bar{z}}^a J_z)\rangle.$$

We see that functional derivatives with respect to A generate correlation functions of the currents. The operator product expansion we wrote in the above can then be expressed in terms of $Z(A)$ as

$$\left(\partial_{\bar{z}}\frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}}\right]\right)Z(A) = \frac{k}{2\pi}\partial_z A_{\bar{z}}Z(A).$$

This is exactly the same equation as that for wave functions for the Chern-Simons theory. In the context of the WZW model, this equation is called the Ward-Takahashi identity since it follows from symmetry of the model.

We can also turn on the A_z component of the gauge field. It is then known that the partition function of the WZW model on a Riemann surface Σ can be decomposed into a finite sum of products of holomorphic and anti-holomorphic parts as,

$$Z(A_z, A_{\bar{z}}) = \sum_{\alpha} \Psi_\alpha(A_{\bar{z}})\bar{\Psi}_\alpha(A_z).$$

Clearly, each Ψ_α should obey the Ward-Takahashi identify, and $\bar{\Psi}_\alpha$ its conjugate. Each of them can be regarded as a wave function for the canonical quantization of the $SU(2)$ WZW model on Σ. In the context of the WZW model, Ψ_α's are called conformal blocks. Thus, we have the correspondence, relating objects in 2-dimensional conformal field theory to the 3-dimensional CS theory,

"conformal blocks in the WZW model on Σ" $\leftrightarrow$ "wave functions of the CS theory on $\mathbf{R} \times \Sigma$".

The relation between the 2d conformal field theory and the 3d theory with coordinate invariance is similar to the AdS/CFT correspondence.

computation

The relation between the CS theory and the WZW model can be used to compute observables in the CS theory. For example, for a closed 3-dimensional manifold M, one can compute its CS partition function by performing the Heegaard splitting and by taking the inner product (Ψ_1, Ψ_2), where $\Psi_{a=1,2}$ are associated to particular handlebodies. Two different handlebodies are related to each other by modular transformations on the boundary Σ, and therefore we can compute this inner product if we know how the modular transformation acts on the Hilbert space $\mathcal{H}_\Sigma$. The description in terms of the WZW model provides this information.

For example, we know that there is only one conformal block on S^2. Therefore,

$$Z_{CS}(S^1 \times S^2) = \mathrm{tr}_{\mathcal{H}_{S^2}} 1 = 1.$$

We can use this to compute $Z_{CS}(S^3)$ since S^3 is related to $S^1 \times S^2$ by the modular transformation $T_{\alpha,\beta} \to T_{\beta,\alpha}$ of the Heegaard decomposition.

Similarly, suppose we want to compute the vacuum expectation values of n Wilson loops on S^3. The S^3 can be separated into two 3d balls, with a common boundary of S^2. Suppose the S^2 cuts each Wilson loops into half. We have S^2 with $2n$ punctures. In fact, they are conformal blocks of a correlation function of $2n$ local operators in the WZW model on S^2. If we move one local operator around another and bring it back to the same location, we should get the same result since that is what it means that the two operators being mutually local. However, conformal blocks can mix with each other under this monodromy transformation. It turns out that, if we cut the Wilson loops in half and perform some monodromy transformation on one side, then one can disentangle knots and links. Thus, if we know how conformal blocks transform into each other under monodromy transformations, one can relate the vacuum expectation values of n Wilson loops to the one of n unknots, the trivial knots. This enables us to compute such expectation values.