Lecture A2

conformal field theory

Killing vector fields

The sphere S^n is invariant under the group $SO(n+1)$. The Minkowski space is invariant under the Poincaré group, which includes translations, rotations, and Lorentz boosts. For a general Riemannian manifold M, take a tangent vector field $\xi = \xi^\mu \partial_\mu$ and consider the infinitesimal coordinate transformation,

$$x^\mu \to x^\mu + \epsilon \xi^\mu, \quad (|\epsilon| \ll 1).$$

Question 1: Show that the metric components $g_{\mu\nu}$ transforms as

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \xi^\rho \partial_\rho\right) + 0(\epsilon^2) = g_{\mu\nu} + \epsilon \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right) + 0(\epsilon^2).$$

The metric is invariant under the infinitesimal transformation by ξ iff $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$. A tangent vector field satisfying this equation is called a Killing vector field.

Question 2: Suppose we have two tangent vector fields, $\xi_a = \xi_a^\mu \partial_\mu$ $(a = 1, 2)$. Show that their commutator

$$[\xi_1, \xi_2] = (\xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu)\partial_\mu$$

is also a tangent vector field. Explain why $\xi_1^\nu \partial_\nu \xi_2^\mu$ is not a tangent vector field in general.

Question 3: Suppose there are two Killing vector fields, ξ_a $(a = 1, 2)$. One can consider an infinitesimal transformation, g_a, corresponding to each of them. Namely,

$$g_a : x^\mu \to x^\mu + \epsilon \xi_a^\mu, \qquad (a = 1, 2).$$

Show that $g_1 g_2 g_1^{-1} g_2^{-1}$ is generated by the commutator, $[\xi_1, \xi_2]$.

If the metric is invariant under infinitesimal transformations generated by ξ_1, ξ_2, it should also be invariant under their commutator, $[\xi_1, \xi_2]$. Thus, the space of Killing vector fields is closed under the commutator – it makes a Lie algebra.

Question 4: What is the Lie algebra of Killing vector fields on S^2?

conformal Killing vector fields

Let us relax the condition somewhat and allow the metric to be scaled as $g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x)$ under transformation, $x \to x + \epsilon\xi$ for some positive-definite function $\Omega(x)$. This means, $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = f(x) g_{\mu\nu}$ for some function $f(x)$. By taking the trace of both sides, one finds $f(x) = \frac{2}{n} \nabla \cdot \xi$ where $n = \dim M$. Thus,

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \frac{2}{n} \nabla \cdot \xi g_{\mu\nu}.$$

A tangent vector field ξ satisfying this equation is called a conformal Killing vector field. Note that $\nabla \cdot \xi = 0$ for a Killing vector field.

Question 5: Count the dimension of the space of conformal Killing vector fields for the n-dimensional Euclidean space and Minkowski space.

In two dimensions, the space of local solutions to the conformal Killing vector field equation is infinite dimensional. Consider a two-dimensional Riemannian manifold M with Euclidean signature (analysis for Minkowskian signature case is the same). In two dimensions, any Riemannian metric is Kähler and one can choose complex coordinates $(z, \bar{z})$ so that the metric becomes $ds^2 = 2g_{z\bar{z}}dzd\bar{z}$. In this metric, the conformal Killing vector equation becomes,

$$\partial_{\bar{z}}\xi^z = 0, \quad \partial_z\xi^{\bar{z}} = 0.$$

Namely, it simply means that ξ^z is holomorphic and $\xi^{\bar{z}}$ is anti-holomorphic. There are infinitely many solutions to this condition. In fact, under any holomorphic coordinate transformation, $w = w(z)$, one finds,

$$g_{w\bar{w}} = \left|\frac{\partial z}{\partial w}\right|^2 g_{z\bar{z}}.$$

Namely, it generates a scale transformation of the metric.

The space of conformal Killing vector fields also makes a Lie algebra. In two dimensions, one can choose $\xi = z^{n+1}\partial_z$ ($n \in \mathbf{Z}$) locally. Then,

$$[\xi_n, \xi_m] = (n-m)\xi_{n+m}. \qquad (1)$$

There is a similar relation for $\bar{\xi} = \bar{z}^{n+1}\partial_{\bar{z}}$.

free massless scalar field in two dimensions

Consider a massless scalar field $\phi(z, \bar{z})$ in two dimensions. Its action is of the form,

$$S = \frac{1}{2\pi}\int_M \sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

so that the equations of motion is given by $\Delta\phi = 0$. The energy-momentum tensor $T_{\mu\nu}$ is defined by

$$T_{\mu\nu} = \frac{2\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2.$$

Note that $T_{\mu\nu}$ is symmetric. It is also traceless, $g^{\mu\nu}T_{\mu\nu} = 0$. It is a reflection of the fact that the action is invariant under the scale transformation, $g_{\mu\nu}(x) \to \Omega(x)g_{\mu\nu}(x)$.

If we use complex coordinates $(z, \bar{z})$, the trace part of the energy-momentum tensor is $T_{z\bar{z}}$. Thus, the only non-zero components are T_{zz} and $T_{\bar{z}\bar{z}}$. The conservation law, $\nabla^\mu T_{\mu\nu} = 0$, then implies that T_{zz} is holomorphic and $T_{\bar{z}\bar{z}}$ is anti-holomorphic. Since the transformation $z \to z + \epsilon z^{n+1}$ is symmetry, there must be a corresponding Nöther charge.

Suppose we start with the Minkowski signature space with coordiantes (t, θ) with the metric $ds^2 = -dt^2 + d\theta^2$. Take θ be periodic so that the space is a cylinder. Now Euclideanize the time coordinate $t = -i\tau$ so that $ds^2 = d\tau^2 + d\theta^2$. If we define $z = e^{\tau + i\theta}$, the past infinity $\tau \to -\infty$ corresponds to $z = 0$ and the fugure infinite $\tau \to +\infty$ corresponds to $z = \infty$. The constant time surface $\tau =$ const. corresponds to $|z| =$ const. In terms of this z coordinate, the Nöther charge for $z \to z + \epsilon z^{n+1}$ is given by

$$L_n = \oint_{z=0}\frac{dz}{2\pi i}z^{n+1}T_{zz}.$$

Here the contour is chosen to surround $z = 0$. Since both z^{n+1} and T_{zz} are holomorphic, L_n is invariant under continuous deformation of the integration contour. The Nöther charge is conserved.

For the free scalar field, the energy-momentum tensor is given by

$$T_{zz} = \frac{1}{2}(\partial_z \phi)^2.$$

Using the operator product expansion,

$$\partial_z \phi(z) \partial_w \phi(w) \sim \frac{1}{(z-w)^2} + 0(1),$$

one obtains

$$T_{zz} T_{ww} \sim \frac{1/2}{(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) T_{ww} + 0(1).$$

Using this, one can derive the commutation relation,

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \qquad (2)$$

with $c = 1$.

The commutation relation of the conformal Killing vectors (1) is also of this form, with $c = 0$. One can think of c as arising from quantum effects. It is called the central charge since it commutes with all the generators L_n. The algebra generated by L_n's is called the Virasoro algebra.

minimal models

The Hilbert space of any conformal field theory (CFT) in two dimensions gives a representation of the Virasoro algebra (2). In fact, there are two copies of the Virasoro algebra, $Vir \oplus Vir$, one for T_{zz} and another for $T_{\bar{z}\bar{z}}$. In this lecture, we assume that the value of the central charge is the same for both Virasoro algebras.

If the CFT is unitary, one should be able to expand its Hilbert space as a sum of unitary representations of $Vir \oplus Vir$. The hermiticity of the energy-momentum tensor implies $L_n^\dagger = L_{-n}$. Since $L_0 + \bar{L}_0$ can be identified with the Hamiltonian for the translation of the time variable t and since

$$[L_0, Ln] = -nL_n,$$

a representation with energy bounded below must be of the highest weight type. Namely, we start with the highest weight state, $|h\rangle$, satisfying

$$L_0|h\rangle = h|h\rangle, \qquad L_n|h\rangle = 0 \qquad (n \geq 1),$$

and build the representation by acting L_{-n}'s. Such a representation is parameterized by the highest weight h. Note that the hermiticity of the energy-momentum tensor implies $L_n^\dagger = L_{-n}$.

It is natural to ask for what value of c, we can build a unitary CFT. For $c < 1$, unitary represenations appear only at discrete values,

$$c = 1 - \frac{6}{m(m+1)}, \qquad (m = 3, 4, 5, ...).$$

For such c, unitary hightest weight representations are parametrized as

$$h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \quad 1 \leq r \leq m-1, \ 1 \leq s \leq r.$$

Let us denote the corresponding highest weight representation by $Vir^{(m)}_{r,s}$. The Hilbert space $Hilb$ of the CFT should take the form,

$$Hilb = \sum_{r,s:r',s'} N_{r,s;r',s'} Vir^{(m)}_{r,s} \oplus Vir^{(m)}_{r',s'}, \qquad (3)$$

where the coefficients $N_{r,s;r',s'}$ are positive integers. Since the vacuum state must be unique, $N_{1,1;1,1} = 1$.

There is a complete classification of $c < 1$ unitary CFT's. The idea is to require the modular invariance discussed in Lecture 9. Consider the character of the Virasoro algebra defined by,

$$\chi_{r,s}(\tau) = \mathrm{tr}_{Vir^{(m)}_{r,s}} e^{2\pi i \tau (L_0 - \frac{c}{24})}.$$

(Do not confuse τ with the imaginary time discussed in the above. It is the modulus of the torus.) It turns out that $\chi_{r,s}(\tau)$'s with $1 \leq s \leq r \leq m-1$ transform into themselves under the modular group action,

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad (ad - bc = 1; a, b, c, d \in \mathbf{Z}).$$

The partition function of the Hilbert space defined by

$$Z(\tau, \bar{\tau}) = \mathrm{tr}_{Hilb} \left(e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c}{24})} \right).$$

The structure of the Hilbert space implies

$$Z(\tau, \bar{\tau}) = \sum_{r,s:r',s'} N_{r,s;r',s'} \chi_{r,s}(\tau) \bar{\chi}_{r',s'}(\bar{\tau}).$$

There is a complete classification of modular invariant partition functions for $c < 1$ unitary conformal field theories.

primary fields

Suppose there is a field $\phi(z, \bar{z})$ which transforms as $\phi (dz)^h (d\bar{z})^{\bar{h}}$. Under infinitesimal coordinate transformation, $z \to z + \epsilon z^{n+1}$, it should transform as,

$$\phi \to \phi + \epsilon \left(z^{n+1} \partial_z + h n z^n \right) \phi(z).$$

Since this coordinate transformation is generated by L_n,

$$[L_n, \phi(z)] = \epsilon \left(z^{n+1} \partial_z + h n z^n \right) \phi(z).$$

This is equivalent to the operator product expansion,

$$T(z)\phi(w) \sim \left(\frac{h}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) \phi(w).$$

The field ϕ is called a primary field and $(h, \bar{h})$ are its (left and right) conformal weights. We see that the energy-momentum tensor $T(z)$ transforms almost as a primary field of weight 2, but there is a slight anomaly due to the term proportional to c.

state-operator correspondence

Suppose the conformal field theory is defined on the geometry $\mathbf{R} \times S^1$, where $\tau \in \mathbf{R}$ corresponds to the Euclideanized time variable and $\in S^1$ parametrizes the spatial section. The Hilbert space is defined at a given time τ. Since the theory is invariant under conformal coordinate transformation, let us introduce

$$z = \exp(\tau + i\theta).$$

In this coordinate, the past infinity is at $z = 0$.

If there is no operator at $z = 0$, the energy-momentum tensor is regular at $z = 0$. This means that

$$L_n(0)1 = \oint_{z=0} \frac{dz}{2\pi i} z^{n+1} T(z) 1 = 0, \quad \text{for } n \geq -1.$$

On the other hand, if we put the primary field $\phi(z, \bar{z})$ of weight h at $z = 0$,

$$L_n(0)\phi(0) = \oint_{z=0} \frac{dz}{2\pi i} z^{n+1} T(z)\phi(0) = 0, \quad \text{for } n \geq 1.$$

Moreover,

$$L_0(0)\phi(0) = h\phi(0).$$

This is just as if we have the highest weight representation of the Virasoro algebra with the highest weight h.

Conversely one can start with a highest weight state $|h\rangle$ in the coordinates (τ, θ) and perform the coordinate transformation in the above to define a primary field $\phi(z, \bar{z})$ at $z = 0$.

This one-to-one correspondence between states and operators is very important in studying conformal field theories.