Lecture 9

Riemann surfaces, elliptic functions

Laplace equation

Consider a Riemannian metric $g_{\mu\nu}$ in two dimensions. In two dimensions it is always possible to choose coordinates (x, y) to diagonalize it as $ds^2 = \Omega(x, y)(dx^2 + dy^2)$. We can then combine them into a complex combination $z = x+iy$ to write this as $ds^2 = \Omega dz d\bar{z}$. It is actually a Kähler metric since the condition $\partial_{[i,}g_{j]\bar{k}} = 0$ is trivial if $i, j = 1$. Thus, an orientable Riemannian manifold in two dimensions is always Kähler.

In the diagonalized form of the metric, the Laplace operator is of the form,

$$\Delta = 4\Omega^{-1}\partial_z\bar{\partial}_{\bar{z}}.$$

Thus, any solution to the Laplace equation $\Delta\phi = 0$ can be expressed as a sum of a holomorphic and an anti-holomorphid function.

$$\Delta\phi = 0 \ \rightarrow \ \phi = f(z) + \bar{f}(\bar{z}).$$

In the following, we assume $\Omega = 1$ so that the metric is $ds^2 = dz d\bar{z}$. It is not difficult to generalize our results for non-constant Ω.

Now, we would like to prove the following formula,

$$\bar{\partial}\frac{1}{z} = -\pi\delta(z),$$

where $\delta(z) = \delta(x)\delta(y)$. Since $1/z$ is holomorphic except at $z = 0$, the left-hand side should vanish except at $z = 0$. On the other hand, by the Stokes theorem, the integral of the left-hand side on a disk of radius r gives,

$$\int_{x^2+y^2\leq r^2} dxdy\ \bar{\partial}\frac{1}{z} = \frac{i}{2}\oint_{|z|=r}\frac{dz}{z} = -\pi.$$

This proves the formula. Thus, the Green function $G(z, w)$ obeying

$$\Delta_z G(z, w) = 4\pi\delta(z - w),$$

should behave as

$$G(z, w) = -\log|z - w|^2 = -\log(z - w) - \log(\bar{z} - \bar{w}),$$

near $z = w$. (In this lecture, log = ln.) Note that the right-hand side is a sum of a holonorphic function and an anti-holomrophic function.

Rational, trigonometric, elliptic

Let us start with the simplest topology in two dimensions – the plane. On the Euclidean plane, $G(z, w) = \log|z - w|^2$ is a solution with the delta-function source at $z = w$. We can add a purely holomorphic or anti-holomorphic function to this, and it still solves the same equation. Compared to three or higher dimensions, none of solutions decay at inifinity. For example, the corresponding solution in three dimensions decays at $1/r$. In n-dimensions, it would be like $1/r^{n-2}$. The limit $n \to 2$ of this gives the logarithmis singularity $\log r$. This is a deep fact and implies, for example, Coleman's theorem that there is no spontaneous breaking of continuous symmetry in quantum field theory in two dimensions.

The next simplest case is the cylinder. Let us periodically identify the x direction with the period 1. We then look for the Green function with the periodicity, $G(z+1,w) = G(z,w+1) = G(z,w)$. This can be satisfied by

$$G(z,w) = -\log|\sin 2\pi(z-w)|^2 .$$

If we periodically identify the cylinde in the y direction, we obtain the torus – the surface of a doughnut. More generally we may twist the cylinder before identification. So, we can impose $(x,y) \sim (x+\theta, y+\beta)$ in addition to $(x,y) \sim (x+1,y)$. In terms of the complex coordinate z, we have

$$z \sim z+1 \sim z+\tau, \quad \tau = \theta + i\beta.$$

Without loss of generality, we assume that β is positive. We can think of the torus as the parallelogram with four vertices $0, 1, \tau, \tau+1$ on the complex planes with its parallel edges identified pair-wisely. The area of the torus is $\beta = \mathrm{Im}\ \tau$.

To find the Green function on the torus, we may consider using the method of images and sum the Green function $G(z,w) = -\log|\sin[2\pi(z-w)]|^2$ over $w \to w + n\tau$ ($n \in \mathbf{Z}$). However, the sum is not convergent. In fact, we can anticipate the problem by noting that the equation,

$$\Delta G(z,w) = \delta(z-w),$$

cannot be compatible with the fact that the torus is compact and without boundary. Integrating both hand side over the torus, the right-hand side gives 1 but the left-hand side would vanish by integration by parts. In order for the Laplace equation to have a solution on a compact space without boundary, the total charge must be zero since the electric flux generated at the source has to go somewhere. To remedy this, we can add a constant negative charge density to cancel the positive charge at $z = w$,

$$\Delta G(z,w) = \delta(z-w) - \frac{1}{\mathrm{Im}\tau}.$$

A solution to this equation is unique up to an additional constant, and it is given by,

$$G(z,w) = -\frac{1}{4\pi}\log\left|\frac{\vartheta_1(z-w|\tau)}{\eta(\tau)}\right| + \frac{1}{2}\frac{(\mathrm{Im}(z-w))^2}{\mathrm{Im}\tau}.$$

Here ϑ_1 is one of the four elliptic theta functions and is given by

$$\begin{aligned}\vartheta_1(z|w) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2} e^{2\pi i\left(n-\frac{1}{2}\right)z} \\ &= 2\sum_{n=1}^{\infty}(-1)^{n-1} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}\sin(2n-1)\pi z, \qquad (1)\end{aligned}$$

where

$$q = e^{2\pi i\tau}.$$

By construction, it satisfies the quasi-periodicity,

$$\vartheta(z+1|\tau) = -\vartheta_1(z|\tau), \quad \vartheta_1(z+\tau|\tau) = -q^{-\frac{1}{8}}e^{-2\pi iz}\vartheta_1(z|\tau).$$

To see that it has a series of zeros at $z = n + m\tau$ ($n, m \in \mathbf{Z}$), it is useful to use the product formula,

$$\vartheta_1(z|\tau) = -iq^{\frac{1}{12}}e^{\pi iz}\eta(\tau)\prod_{n=1}^{\infty}(1-q^ne^{2\pi iz})(1-q^{n-1}e^{-2\pi iz}),$$

where $\eta(\tau)$ is the Dedekint eta-function,

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n).$$

There are three other theta functions,

$$\begin{aligned}
\vartheta_0(z|\tau) &= q^{-\frac{1}{24}}\eta(\tau)\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}e^{2\pi iz})(1-q^{n-\frac{1}{2}}e^{-2\pi iz}),\\
\vartheta_2(z|\tau) &= q^{\frac{1}{12}}e^{\pi iz}\eta(\tau)\prod_{n=1}^{\infty}(1+q^ne^{2\pi iz})(1+q^{n-1}e^{-2\pi iz}),\\
\vartheta_3(z|\tau) &= q^{-\frac{1}{24}}\eta(\tau)\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}e^{2\pi iz})(1+q^{n-\frac{1}{2}}e^{-2\pi iz}). \qquad (2)
\end{aligned}$$

In particular, ϑ_3 is called the Jacobi theta function. The other three theta functions are obtained from the Jacobi theta function by shifting z by $1/2$ and $\tau/2$.

Modular invariance

We have regarded the torus as the quotient of $\mathbf{C}$ with the metric $ds^2 = dzd\bar{z}$ by the lattice $\{n + m\tau : n, m \in \mathbf{Z}\}$. Equivalently, we can consider a fixed lattice and modify the metric as a function of τ. Consider coordinates (u, v) on $\mathbf{R}^2$ with the periodicity $(u, v) \sim (u + n, v + m)$ ($n, m \in \mathbf{Z}$). Namely, we are considering the square lattice. To reproduce the periodicity $z \sim z + 1 \sim z + \tau$, we can write $z = u + \tau v$. The metric in terms of the coordinates (u, v) then becomes

$$ds^2 = |du + \tau dv|^2.$$

Physics should be independent of the choice of coordinates we make. We can change coordinates as

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

This does not change the lattice structure if the matrix is in $SL(2,\mathbf{Z})$. Consider two examples of $SL(2,\mathbf{Z})$ transformations:

(1) $(u, v) \to (u + v, v)$: This changes the metric as

$$ds^2 = |du + (\tau + 1)dv|^2.$$

This means that the metrics with τ and $(\tau + 1)$ are related to each other by the coordinate transformation.

(2) $(u,v)\to(v,-u)$: This changes the metric

$$ds^2 = |\tau|^2\left|du - \frac{1}{\tau}dv\right|^2.$$

This means that the metrics with τ and $-1/\tau$ are related to each other by the coordinate transformation adn the overall rescaling. The Laplace equation is invariant under the rescaling of the metric, so its solution must have this symmetry too.

Question 1: Show that, in two dimensions, the Laplace equation with the unit source, $\Delta G(z,w) = \delta(z-w)$ is invariant under an arbitrary rescaling of the metric $g_{\mu\nu}\to\Omega(z,\bar{z})g_{\mu\nu}$. Here the delta-function is normalized with respect to the volume form,

$$\int \sqrt{g}\;\delta(z-w) = 1.$$

It is known that the two transformation $(u,v)\to(u+v,v)$ and $(v,-u)$ generate the whole $SL(2,\mathbf{Z})$ group. Its action on τ can be seen as

$$\tau\to\frac{a\tau+b}{c\tau+d}.$$

This is called the modular transformation. As expected, the ingredients of the Green function on the torus transforms nicely under the modular transformation as,

$$\eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau),\quad \eta\left(-1/\tau\right) = \sqrt{-i\tau}\eta(\tau),$$

and

$$\vartheta_1(z|\tau+1) = e^{\frac{2\pi i}{8}}\vartheta_1(z|\tau),\quad \vartheta_1\left(z/\tau|-1/\tau\right) = e^{-\frac{2\pi i}{8}+\frac{2\pi i z^2}{2\tau}}\tau^{-\frac{1}{2}}\vartheta_1(z|\tau).$$

The parameter τ specifies the complex structure of torus as it determines its complex coordinate $z = u+\tau v$ with respect to the fixed real coordinates (u,v). Two different τ's related to each other under the modular transformation describe the same complex structure. The space of complex structures on the torus is called the moduli space of the torus, and it can be identifies as the upper half-plane of the τ-space, modulo $SL(2,\mathbf{Z})$.

Elliptic integral

With the elliptic theta function, we can define Weirstrass' elliptic function as

$$\mathcal{P}(z) = -\frac{\partial^2}{\partial z^2}\log\vartheta_1(z|\tau) - 2\eta_1(\tau),$$

where

$$\eta_1(\tau) = 2\pi i\frac{\partial}{\partial\tau}\log\eta(\tau).$$

This function has a double pole at $z=0$ and is doubly-periodic,

$$\mathcal{P}(z) = \mathcal{P}(z+1) = \mathcal{P}(z+\tau).$$

Its derivative $\mathcal{P}'(z)$ has a triple pole at $z=0$ and is also doubly-periodic. A doubly-periodic function on the torus is called an elliptic function. It can be shown that any elliptic function is a rational function of $\mathcal{P}(z)$ and $\mathcal{P}'s(z)$.

They satisfies the relation,

$$(\mathcal{P}'(z))^2 = 4(\mathcal{P}(z) - e_1)(\mathcal{P}(z) - e_2)(\mathcal{P}(z) - e_3),$$

where e_1, e_2, e_3 are some functions of τ, given by

$$e_1 = \mathcal{P}\left(\frac{1}{2}\right), \quad e_2 = \mathcal{P}\left(\frac{\tau}{2}\right), \quad e_3 = \mathcal{P}\left(\frac{1}{2} + \frac{\tau}{2}\right).$$

The above relation between $\mathcal{P}'$ and $\mathcal{P}$ means that $\mathcal{P}'$ is a function of $\mathcal{P}$ with branch points at $\mathcal{P} = e_1, e_2, e_3$. We can have two branch cuts, one connecting e_1 and e_2, for example, and another going from e_3 to the infinity. One can see that the covering space is topologically equivalent to the torus.

The equation can also be expressed as,

$$z = \int^{\mathcal{P}} \frac{d\mathcal{P}}{2\sqrt{(\mathcal{P}(z) - e_1)(\mathcal{P}(z) - e_2)(\mathcal{P}(z) - e_3)}} + \text{const.}$$

The integral on the right-hand side is of the form known as elliptic integral. Thus, the Weierstrass $\mathcal{P}$-function can be regarded as the inverse of the elliptic integral. It is because of this historical origin that doubly-periodic functions are called elliptic functions. Historically, elliptic integrals were studied to compute the arc length of an ellipse. It occured later to Abel and Jacobi that its inverse has the double-periodicity $z \sim z + 1 \sim z + \tau$.

Riemann surfaces of higher genera

The torus can be regarded as a set of solution to the equation,

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3).$$

This can be solved by setting $y = \mathcal{P}'(z)$ and $x = \mathcal{P}(z)$, with z being the doubly-periodic coordinate. This can be generalized. For example, one can consider the equation,

$$y^2 = \prod_{i=1}^{2g+1} (x - e_i).$$

We can think of y as a function of x with $(g+1)$ branch cuts, one of which extends to the infinity in the x-plane. The covering space is then a surface with g handles. It is a special example of Riemann surfaces of genus g, called the hyper-elliptic surface. The torus has genus $g = 1$.

The torus as a complex manifold is parametrized by the modulus τ. For a general Riemann surface Σ_g of genus $g > 1$, the moduli space $\mathcal{M}_g$ is a complex manifold of dimensions $(3g - 3)$. To understand $\mathcal{M}_g$, it is useful to consider period integrals. It is easy to see that $b_1(\Sigma_g) = 2g$. We can choose a basis $H_1(\Sigma_g)$ so that for each handle, we have two generators α_i, β^i $(i = 1, ..., g)$ and that they intersect with each other as,

$$\alpha_i \cap \alpha_j = \beta^i \cap \beta^j = 0, \quad \alpha_i \cap \beta^j = \delta_i^j.$$

With respect to this basis, we can choose a basis of $H^{1,0}(\Sigma_g)$ as $\{\omega_i\}$ so that

$$\int_{\alpha_i} \omega_j = \delta_j^i.$$

The period matrix is defined as

$$\Omega_{ij} = \int_{\beta^i} \omega_j.$$

There is a one-to-one map from the moduli space $\mathcal{M}_g$ to Ω_{ij}, namely we can use Ω_{ij} to distinguish complex structures of Σ_g. Unfortunately, the space of Ω_{ij} is too big; it is a symmetric matrix, so it has $\frac{1}{2}g(g+1)$ components while $\dim\mathcal{M}_g = 3g-3$. This raises the question on how to characterize the image of $\mathcal{M}_g$ in the space of $g \times g$ symmetric matrices. This so-called Schottkey problem was found by using the theory of integrable systems.

There is an analogue of the theta functions at higher genera, called the Riemann theta function,

$$\Theta(\vec{z}|\Omega) = \sum_{\vec{n}\in\mathbf{Z}^g} \exp\left(2\pi i\left(\frac{1}{2}\vec{n}^t\Omega\vec{n} + \vec{n}\cdot\vec{z}\right)\right).$$

We can build the theory of holomorphic functions and sections of various line bundles over Σ_g using the Riemann theta function.