**Lecture 8**

*supersymmetry and index theorems*

bosonic sigma model

Let us consider a dynamical system describing a motion of a particle in a Riemannian manifold $M$. The motion is a map $\phi : \mathbf{R} \to M$. By choosing coordinates $(\phi^\mu)_{\mu=1,\dots,n}$ on $M$, we can write the Lagrangian,

$$L = \frac{1}{2} g_{\mu\nu}(\phi)\dot{\phi}^\mu \dot{\phi}^\nu,$$

where $g_{\mu\nu}$ is the metric on $M$ and $\dot{\phi} = d\phi/dt$. Note that this is independent of the choice of coordinates. The momentum conjugate to $\phi^\mu$ is given by

$$p_\mu = \frac{\partial L}{\partial \dot{\phi}^\mu} = g_{\mu\nu}\dot{\phi}^\nu.$$

The Hamiltonian is then,

$$H = p_\mu \dot{\phi}^\mu - L = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu.$$

To quantize this system, we consider the Hilbert space consisting of square integrable functions $\Psi(\phi)$ over $M$. We can then define the inner product,

$$(\Psi_1, \Psi_2) = \int d^n\phi \sqrt{g} \bar{\Psi}_1 \Psi_2.$$

The multiplication of the coordinates $\phi^\mu$ defines hermitian operators. We can also define the momentum components $p_\mu$ as hermitian operators as

$$p_\mu = -\sqrt{-1} g^{-\frac{1}{4}} \frac{\partial}{\partial \phi^\mu} g^{\frac{1}{4}},$$

so that

$$[\phi^\mu, p_\nu] = \sqrt{-1}\delta^\mu_\nu.$$

Similarly, there is some operator ordering ambiguity in defining the quantum version of the $H$. Requiring invariance under coordiante change, we use

$$H = -\frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \phi^\mu} \left( g^{\mu\nu} \sqrt{g} \frac{\partial}{\partial \phi^\nu} \right).$$

This is nothing but the (minus of) Laplace operator on $M$.

The Hamiltonian $H$ can be used to write the Schrödinger equation,

$$i \frac{\partial}{\partial t} \Psi = H\Psi,$$

which can be formally integrated as

$$\Psi(t) = e^{-itH} \Psi(t=0).$$

For our purpose, it is often convenient to analytically continue the time variable $t$ to pure imaginary value $t = -i\tau$ with $\tau \in \mathbf{R}$. Thus,

$$\Psi(\phi, \tau) = e^{-\tau H}\Psi(\phi, \tau = 0).$$

We can introduce the heat kernel $G(\phi_1, \phi_2; \tau)$ to express this as

$$\Psi(\phi_1, \tau) = \int d^n\phi_2 G(\phi_1, \phi_2; \tau)\Psi(\phi_2).$$

The heat kernel obeys the Schrödinger equation (or more precisely the diffusion equation),

$$\frac{\partial}{\partial\tau}G = -HG$$

with the initial condition,

$$G(\phi_1, \phi_2; \tau = 0) = \delta(\phi_1 - \phi_2).$$

To understand the energy spectrum $\{\epsilon_0, \epsilon_1, ...\}$ of the Hamiltonian, it is useful if we can evaluate $\mathrm{tr} e^{-\tau H}$ as it gives,

$$\mathrm{tr} e^{-\tau H} = \sum_{i=0}^{\infty} e^{-\tau\epsilon_i},$$

where tr is over the Hilbert space (*i.e.* the space of square integrable functions over $M$). Using the heat kernel, we can write it as

$$\mathrm{tr} e^{-\tau H} = \int d^n\phi G(\phi, \phi; \tau).$$

In the Feynman path integral formulation of quantum mechanics, the heat kernel $G(\phi, \phi'; \tau)$ is expressed as

$$G(\phi_1, \phi_2; \tau) = \int_{\phi(\tau)=\phi_1;\phi(0)=\phi_2} \mathcal{D}\phi \exp\left(-\int_0^\tau d\tau' L(\dot{\phi}(\tau'), \phi(\tau'))\right).$$

Combining this with the above expression for $\mathrm{tr} e^{-\tau H}$, we find that

$$\mathrm{tr} e^{-\tau H} = \int_{\phi(\tau)=\phi(0)} \mathcal{D}\phi e^{-\int_0^\tau d\tau' L}.$$

supersymmetric sigma-model

We introduce fermionic coordinates $\psi^\mu$ for the tangent space $T_\phi M$ at $\phi$. The Lagrangian

$$\begin{aligned} L &= \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{\sqrt{-1}}{2}g_{\mu\nu}\bar{\psi}^\mu\left(\frac{d}{dt}\psi^\nu + \Gamma^\nu_{\rho\sigma}\dot{\phi}^\rho\psi^\sigma\right) \\ &\quad + \frac{1}{4}R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma. \end{aligned} \tag{1}$$

As I advertised earlier, the Hilbert space is the space of differential forms $\Omega(M)$ (we require them to be integrable with respect to the inner product on $\Omega(M)$). The fermions obey the anti-commutation relations,

$$\{\psi^\mu, \psi^\nu\} = 0,\ \{\bar{\psi}^\mu, \bar{\psi}^\nu\} = 0,\ \{\psi^\mu, \bar{\psi}^\nu\} = g^{\mu\nu}.$$

The vacuum $|0\rangle$ in the ferimion Fock space is annihilated by all the $\psi$'s. We identify it as the 0-form. Other forms are generated by multiplying $\bar{\psi}$'s, as $\bar{\psi}^\mu|0\rangle, \bar{\psi}^\mu\bar{\psi}^\nu|0\rangle, \ldots$.

The supercharges $Q, \bar{Q}$ are identified with the exterior derivative operator $d$ and its conjugate $\delta \sim *d*$, so that $H = \{Q, \bar{Q}\}$ gives the Laplace-Beltrami operator.

**Witten index**

Let $F$ be the operator that counts the degree of forms. We can also call it as the fermion number. We can use it to define the fermion number parity $(-1)^F$, which gives $+1$ for bosonic states (even forms) and $-1$ for fermoinic states (odd forms). Note that $(-1)^F Q = -Q(-1)^F$ and similarly for $\bar{Q}$. Since $Q$ and $\bar{Q}$ commute with $H$, it maps a bosonic energy eigenstate to a fermionic energy eigenstate and vice versa. Moreover, since $H = \{Q, \bar{Q}\}$, the map is bijection (one-to-one and onto) for states with non-zero energies. The situation is different for states with $H = 0$; the numbers of bosonic and fermionic states may be different. The difference is called the Witten index. The Witten index is a convenient quantity since it is invariant under continous deformations of the system, provided that the spectrum does not contain continuous bands.

For the supersymmetric sigma model we are discussing, the Witten index is given by

$$\mathrm{tr}(-1)^F e^{-\beta H} = \sum_{p=0}^{n} (-1)^p b_p = \chi(M).$$

**Grassmannian integral**

Let us evaluate $\chi(M)$ using the Feynman path integral. To do so, we need to formulate the path integral for the fermionic variables $\psi, \bar{\psi}$. This is done by introducing Grassmannian numbers.

A Grassmannian number $\theta$ is nilpotent $\theta^2 = 0$, and its integral is defined like a differentiation,

$$\int d\theta 1 = 0, \quad \int d\theta\theta = 1.$$

More generally, if we have a function $f(\theta) = a + b\theta$, wehre $a$ and $b$ are ordinary numbers (since $\theta^2 = 0$, the most general function is a linear function), we have

$$\int d\theta f(\theta) = b.$$

A nice thing about this definition is that we can perform the integration by parts,

$$\int d\theta \frac{\partial}{\partial\theta} f(\theta) = \int d\theta b = 0.$$

Similarly, we an show that

$$\int d\theta f(\theta) \frac{\partial}{\partial\theta} g(\theta) = \pm \int d\theta \left[\frac{\partial}{\partial\theta} f(\theta)\right] g(\theta),$$

where the sign on the right-hand side depends on whether $b\theta = \theta b$ (choose $+$) or $b\theta = -\theta b$ (choose $-$).

Question 1: For a bosonic variable $x$, achange of variables $x \to y$ with $x = f(y)$ transforms an integral as

$$\int dx G(x) = \int dy |f'(y)| G(f(y)).$$

Show that, for a Grassmannian variable $\theta$, a change of variables $\theta \to \rho$ with $\theta = f(\rho)$ gives

$$\int d\theta G(\theta) = \int d\rho \frac{1}{f'(\rho)} G(f(\rho)).$$

This definition fits well with the interpretation of the supersymmetric sigma-model in terms of differential forms. Consider the top form $\omega \in \Omega^n(M)$. We can write the integral of $\omega$ over $M$ as,

$$\int_M \omega = \int d^n\phi d^n\bar{\psi} \omega_{\mu_1\mu_2\cdots\mu_n}(\phi) \bar{\psi}^{\mu_1} \bar{\psi}^{\mu_2} \cdots \bar{\psi}^{\mu_n}.$$

Question 2: Use the result of Question 1 to show that the right-hand side of the above is independent of the choice of coordinates $\phi^\mu$.

**Gauss-Bonnet theorem**

Now we are ready to evaluate the Witten index $\mathrm{tr}(-1)^F e^{-\beta H}$ using the Feynman path integral. As in the computation $\mathrm{tr} e^{-\beta H}$ in the case of the bosonic sigma-model, we consider a sum over periodic paths, but now in $TM$ not in $M$.

It is important to note that we require periodic boundary condtion for $\psi$ and $\bar{\psi}$ as $\tau \to \tau + \beta$ because of the insertion of $(-1)^F$. To see this, let us look at the following quantity, $\mathrm{tr}(-1)^F e^{-\beta H} \mathcal{O} \psi(0)$ for some operator $\mathcal{O}$. By using the cyclicity of the trace, we see that this is equal to $-\mathrm{tr}(-1)^F e^{-\beta H} \psi(\beta) \mathcal{O}$, where the minus sign comes from the exchange of $\psi$ with $(-1)^F$. However, we should also note that $\mathcal{O}$ must be a fermionic operator – otherwise the trace would vanish. Thus, we get an extra minus sign when we exchange $\psi$ and $\mathcal{O}$. The end result is that $\psi(0) = \psi(\beta)$ in the trace.

Now we are ready to evaluate the path integral, with periodic boundary conditions for both $\phi$ and $\psi, \bar{\psi}$. The main idea is use of the fact that the answer is independent of $\beta$ and look at the limit of $\beta \to 0+$. Define $t = \beta\tau$ keeping the periodicity of $\tau$ to be 1. Rescale $\psi \to \beta^{-1/4}\psi$, and we obtain,

$$L = \frac{1}{2\beta^2} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu + \frac{\sqrt{-1}}{2\beta^{\frac{3}{2}}} g_{\mu\nu} \bar{\psi}^\mu \left( \frac{d}{dt} \psi^\nu + \Gamma^\nu_{\rho\sigma} \dot{\phi}^\rho \psi^\sigma \right) + \frac{1}{4\beta} R_{\mu\nu\rho\sigma} \psi^\mu \psi^\nu \bar{\psi}^\rho \bar{\psi}^\sigma.$$

The action is then

$$\begin{aligned} S &= \beta \int_0^1 d\tau L \\ &= \iint_0^1 d\tau \Big[ \frac{1}{2\beta} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu + \frac{\sqrt{-1}}{2\beta^{\frac{1}{2}}} g_{\mu\nu} \bar{\psi}^\mu \left( \frac{d}{dt} \psi^\nu + \Gamma^\nu_{\rho\sigma} \dot{\phi}^\rho \psi^\sigma \right) + \frac{1}{4} R_{\mu\nu\rho\sigma} \psi^\mu \psi^\nu \bar{\psi}^\rho \bar{\psi}^\sigma \Big]. \quad (2) \end{aligned}$$

In the limit of $\beta \to 0$, the only configurations that can contribute to the path integral are are those with $\phi$ and $\psi$ being constant since any non-constant configuration would make the action infinite in the limit. Thus, the path integral reduces to an integral over constant $\phi$ and $\psi$ as,

$$\mathrm{tr}(-1)^f e^{-\beta H} = \int \frac{d^n\phi d^n\psi d^n\bar{\psi}}{(2\pi)^{n/2}} e^{-\frac{1}{4}R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma}.$$

This would be 0 if $n = \mathrm{dim}M$ is odd. If $n = 2m$, we can evaluate the Grassmannian integral to find

$$= \frac{(-1)^m}{2^{2m}m!\pi^m} \int d^n\phi\sqrt{g}\epsilon^{\mu_1\nu_1\cdots\mu_m\nu_n}\epsilon^{\rho_1\sigma_1\cdots\rho_m\sigma_m}R_{\mu_1\nu_1\rho_1\sigma_1}\cdots R_{\mu_m\nu_m\rho_m\sigma_m}.$$

This gives a path integral proof of the Gauss-Bonnet theorem.

**Morse theory**

We can also turn on a potential $h(\phi)$ in a supersymmetric way,

$$\Delta L = -\frac{1}{2}g^{\mu\nu}\frac{\partial h}{\partial\phi^\mu}\frac{\partial h}{\partial\phi^\nu} - \frac{1}{2}\frac{\partial^2 h}{\partial\phi^\mu\partial\phi^\nu}\psi^\mu\psi^\nu.$$

In this case, the path integral localizes at teh minima of $|\partial h/\partial\phi|^2$, namely at $\partial h/\partial\phi^\mu = 0$. By using a scaling argument similar to the one in the above, we find

$$\mathrm{tr}(-1)^F e^{-\beta H} = \sum_{\phi_0:\partial h(\phi_0)=0} (-1)^{\#(\text{negative eigenvalues of } \partial^2 h)}.$$

In fact, to the leading order in the $\hbar$ expansion, we find

$$b_p = \mathrm{dim}H^p(M) = \#\left\{\phi_0 : \frac{\partial h}{\partial\phi^\mu} = 0,\ \#(\text{negative eigenvalues}) = p\right\}.$$

This can be refined by incorporating instanton corrections.

**Dirac operator**

In the supersymmetric sigma-model discussed in the above, there are two supercharges $Q$ and $\bar{Q}$. We can reduce the amount of supersymmetry by half, by setting $\bar{\psi} = \psi$. In this case, we have only $Q = \bar{Q}$. The Lagrangian is simplified as

$$L = \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{\sqrt{-1}}{2}g_{\mu\nu}\psi^\mu D_t\psi^\nu.$$

The quantization gives the anti-commutatition relation,

$$\{\psi^\mu, \psi^\nu\} = g^{\mu\nu}.$$

This is the Clifford algebra. It is known that the Clifford algebra has a unique non-trivial irreducible representation and that is the representation in terms of the Dirac matrices $\gamma^\mu$. Thus, the Hilbert space of this model is the space of spinor fields on $M$.

In this case, $Q = \gamma^\mu D_\mu$.

The Witten index in this case is called the Dirac genus. The sigma-model path integral shows that it is given by the $\hat{A}$ polynomial,

$$\mathrm{ind}(\gamma^\mu D_\mu) = \int_M \hat{A},$$

where

$$\hat{A} = \prod_{i=1}^{m} \frac{x_i/2}{\sinh(x_i/2)},$$

and $x_i$ are eigenvalues of the Riemann curvature $(i = 1, ..., m; \mathrm{dim} M = 2m)$.

**Dolbeault index**

If $M$ is a Kähler manifold, we can double the number of supercharges, $\partial, \bar{\partial}, \partial^\dagger, \bar{\partial}^\dagger$. The arithmetic genus of the manifold is defined by

$$\mathrm{index}(\bar{\partial}) = \sum_q (-1)^q \mathrm{dim} H_{\bar{\partial}}^{(0,q)}(M).$$

The path integral computation is exactly the same as that for the Dirac genus, and it relates the index to the Todd class,

$$td(M) = \prod_i \frac{x_i}{1 - e^{-x_i}}.$$

These index theorems are interesting in that they relate global quantitizes such as the Euler characteristic to integrals of curvatures (local quantities).