Lecture 7 *characteristic classes*

In the previous lectures, we have seen cases where fiber bundles are characterized by integers. For example, monopole bundles on S^2 are classified by the monopole number n which tells us how the $U(1)$ fibers over the upper hemisphere and the lower hemisphere are glued together. Generally speaking, for a vector bundle on a manifold M, a characteristic class associates a cohomology class of M.

Invariant Polynomials

Characteristic classes are constructed as polynomials of the curvature $F = dA + A \wedge A$. Under gauge transformation, F transforms as $F \to \Omega^{-1} F \Omega$, where Ω is a map from the manifold M to the gauge group (structure group) G. In the following, we consider the cases where $G = U(k)$ and $SO(2r)$. To construct characteristic classes, we need to introduce invariant polynomials of matrixes. We look for a function $P(X)$ of a matrix X that is invariant under the conjugation, $P(\Omega^{-1} X \Omega) = P(X)$. We consider two cases:

(1) X is a $k \times k$ hermitial matrix and $\Omega \in U(k)$. This will be used when E is a complex vector bundle.

(2) X being a $2r \times 2r$ real anti-symmetric matrix and $\Omega \in SO(2r)$. This will be used when E is a real vector bundle.

Examples of invariant polynomials are $\mathrm{tr} X^m$ $(m = 1, 2, ...)$ and $\det X$. In fact we can use these to construct a nice basis. The following two are particularly useuful (we are using the notation in the case of $G = U(k)$):

(1) $\sigma_i(X)$ defined by

$$\det(1 + tX) = 1 + t\sigma_1(X) + t^2 \sigma_2(X) + \cdots + t^k \sigma_k(X).$$

(2) $s_i(X)$ defined by

$$s_i(X) = \mathrm{tr} X^i, \quad (i = 1, ..., k).$$

They are related to each other by Newton's formula,

$$s_1 = \sigma_1, \ s_2 = \sigma_1^2 - \sigma_2, \ s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3, ...$$

We can also express the invariant polynomials in terms of eigenvalues. If X is a hermitian matrix, we can diagonalize it with eigenvalues $x_1, ..., x_k$. Then,

$$\prod_{i=1}^{k} (1 + t x_i) = 1 + t\sigma_1(x) + \cdots + t^k \sigma_k(x).$$

Similarly,

$$s_i(x) = \sum_{j=1}^{k} (x_j)^i.$$

Characteristic Classes

If $P_i(X)$ is an invariant polynomial of degree i, we can use the curvature 2-form F to define a $2i$-form $P_i(F)$. By construction, it is invariant under the gauge transformation, $F \to \Omega^{-1} F \Omega$. It is also a closed form. To see this, we first note that F satisfies the Bianchi identity,

$$dF + [A, F] = 0.$$

Question 1: Verfity the above identity. To show this, one has to use the Jacobi identity

$$[a, [b, c]] + \text{cyclic perm.} = 0.$$

Using this, we find

$$\begin{aligned} d\mathrm{tr}F^i &= \mathrm{tr}\left\{dFF^{i-1} + FdFF^{i-1} + \cdots F^{i-1}dF\right\} \\ &= \mathrm{tr}\left\{(dF + [A, F])F^{i-1} + \cdots + F^{i-1}(dF + [A, F])\right\} \quad (1) \\ &= 0 \quad (2) \end{aligned}$$

Thus, $P_i(F)$ is potentially a non-trivial element of $H^{2i}(M)$.

Moreover, $P_i(F)$ is invariant under continuos deformation of the gauge field A as an element of $H^{2i}(M)$. Suppsoe we change $A \to A + \eta$ with η being an infinitesimal one-form. Note that, although A transforms inhomogeneously as $A \to \Omega^{-1} A \Omega + \Omega^{-1} d\Omega$, the one-form transforms homogeneously, $\eta \to \Omega^{-1} \eta \Omega$.

Under this deformation, F changes by $\delta F = d\eta + [A, \eta]$. Therefore,

$$\begin{aligned} \delta\mathrm{tr}F^i &= \mathrm{tr}\left\{(d\eta + [A, \eta])F^{i-1} + \cdots + F^{i-1}(d\eta + [A, \eta])\right\} \\ &= i\mathrm{tr}\left\{(d\eta + [A, \eta])F^{i-1}\right\} \\ &= i\mathrm{tr}\left\{d\eta F^{i-1} + \eta dFF^{i-2} \cdots + \eta F^{i-2}dF\right\} \quad (3) \\ &= id\mathrm{tr}\left(\eta F^{i-1}\right). \quad (4) \end{aligned}$$

Since both η and F transform homogeneously under the gauge transformation, $\mathrm{tr}(\eta F^{i-1})$ is a well-defiend $(2i-1)$-form. Thus, under any infinitesimal deformation, $P_i(F)$ changes by an exact form. Thus, $P_i(F)$ depends only on the type of the bundle E and not on a specific type of the connection A on E. For this reason, we sometime write a characteristic class as a function of E.

Chern classes, Chern characters, etc

Among characteristic classes for an $U(n)$ bundle, we have the Chern classes and the Chern characters.

The **Chern classes**, $c_i \in H^{2i}(M)$ $(i = 0, 1, 2..., k)$, are defined by

$$\det\left(1 + \frac{\sqrt{-1}}{2\pi}F\right) = c_0 + c_1 + c_2 + \cdots.$$

For example,

$$c_0 = 1, \; c_1 = \frac{\sqrt{-1}}{2\pi}\mathrm{tr}F, \; c_2 = -\frac{1}{8\pi^2}\left(\mathrm{tr}F \wedge \mathrm{tr}F - \mathrm{tr}F \wedge F\right), \; \cdots.$$

If the holonomy is in $SU(k) \in U(k)$, we have a trivial first Chern class $c_1 = 0$.

The sum $c = c_0 + c_1 + c_2 + \cdots$ is called the total Chern class. One of the important properties of the Chern classes is that it behaves nicely when we take a direct sum of vector bundles $E_1, E_2 \to E_1 \oplus E_2$ as,

$$c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2).$$

On the other hand, it does not behave nicely under the direct product $E_1 \otimes E_2$.

The **Chern characters**, $ch_i(F) \in H^{2i}(M)$ $(i = 0, 1, 2, \cdots)$, are defined by

$$ch_i(F) = \frac{1}{i!}\mathrm{tr}\left(\frac{\sqrt{-1}F}{2\pi}\right)^i.$$

We can also write it as

$$ch(F) = ch_0 + ch_1 + \cdots = \mathrm{tr}\exp\left(\frac{\sqrt{-1}F}{2\pi}\right).$$

The Chern characters behave nicely under both the direct sum and direct product as,

$$\begin{aligned} ch(E_1 \oplus E_2) &= ch(E_1) + ch(E_2), \\ ch(E_1 \otimes E_2) &= ch(E_1) \wedge ch(E_2). \end{aligned} \tag{5}$$

Sometime we encounter other characteristic classes, such as Todd classes, Hirzebuch L-polynomials, and $\hat{A}$ polynomials. They correspond to different basis' of invariant polynomials. To desribe these, use eigenvalues $x_1, ..., x_k$ of $\frac{\sqrt{-1}}{2\pi}F$. For example, the total Chern classes can be expressed as

$$c(F) = \det\left(1 + \frac{\sqrt{-1}}{2\pi}F\right) = \prod_{i=1}^{k}(1 + x_i).$$

(Tangentially, it is interesting to note that the right-hand side takes the form $\prod_i c(L_i)$, where L_i is a line bundle with a curvatuve given by x_i and $c(L_i) = 1 + x_i$. Thus, as far as the Chern classes are concerned, the bector bundle E behaves as a direct sum of the line bundles $L_1 \oplus L_2 \oplus \cdots \oplus L_k$. This phenomenon is called the splitting principle.)

Using this notation, the Todd classes are defined by,

$$td(E) = \prod_i \frac{x_i}{1 - e^{-x_i}},$$

the L-polynomials are defined by,

$$L(E) = \prod_i \frac{x_i}{\mathrm{tanh} x_i},$$

and the $\hat{A}$-polynomials are defined by,

$$\hat{A}(E) = \prod_i \frac{x_i/2}{\sinh(x_i/2)}.$$

Chern number

Remarkably, the Chern classes and the Chern characters are integral. That means that if we integrate, say, $c_i(E)$ over any $2i$-cycle in M with integer coefficients, we find an integer that is independent of the choice of the connection of E. If $2k \geq n$, we can integrate $c_n(F)$ over the entire manifold M and get the Chern number. Let us compute Chern numbers in some examples.

(1) Consider the monopole bundle over S^2. It has the $U(1)$ gauge field A. Let us denote the northen and southern hemispheres of S^2 as $H_\pm$, and the gauge fields on them as $A_\pm$. For the monopole bundle with n monopole charge, the gauge field transforms as

$$A_- = A_+ + n d\phi$$

across the equator, where ϕ is the longitude of S^2. We can then evaluate the Chern number as

$$\begin{aligned} C_1 &= \int_{S^2} c_1 \\ &= \frac{-1}{2\pi}\left(\int_{H_+} A_+ + \int_{H_-} A_-\right) \\ &= \frac{-1}{2\pi}\int_{S^1}(A_+ - A_-) = \frac{1}{2\pi}\int_0^{2\pi} n d\phi = n. \end{aligned} \tag{6}$$

Thus, the monopole number is the first Chern number in this case.

(2) Consider an $SU(2)$ bundle over S^4. We can then consider the second Chern number,

$$C_2 = \int_{S^4} c_2 = \frac{1}{8\pi^2}\int_{S^4} \mathrm{tr} F \wedge F.$$

We again split S^4 into $H_\pm$ such that $H_+ \cap H_- = S^3$. Over S^3, the gauge field transforms as

$$A_- = \Omega^{-1} A_+ \Omega + \Omega^{-1} d\Omega.$$

When we integrate $\mathrm{tr} F_\pm \wedge F_\pm$ over $H_\pm$, we note that the integrand can be written as

$$\mathrm{tr} F \wedge F = d\,\mathrm{tr}\left(A dA + \frac{2}{3}A^3\right).$$

Note that this does not mean that $\mathrm{tr} F \wedge F$ is an exact form since the right-hand side, called the Chern-Simons form, is not necessarily globally defined over S^4. Thus,

$$\begin{aligned} C_2 &= \int_{S^4} c_2 \\ &= \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\left(A_+ dA_+ + \frac{2}{3}A_+^3 - A_- dA_- - \frac{2}{3}A_-^3\right) \\ &= \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(\Omega^{-1} d\Omega)^3. \end{aligned} \tag{7}$$

The gauge transformation matrix Ω is a map from S^3 to $SU(2)$. Since the group $SU(2)$ is diffeomorphic to S^3 as a manifold, we can think of it as a map from S^3 to itself. Such a map can be classified by its winding number, which turns out to be the same as the second Chern number in the above.

Euler class, Poltryagn classes

Let us turn to real vector bundles. Suppose X is a $2r \times 2r$ real and anti-symmetrix matrix. In this case, in addtion to tr and det, we can consider one more way to construct an invariant polynomial. That is the Pfaffian,

$$Pf(X) = \frac{(-1)^r}{2^r r!} \epsilon^{i_1 j_1 i_2 j_2 \cdots i_r j_r} X_{i_1 j_1} X_{i_2 j_2} \cdots X_{i_r j_r}.$$

Note that, for antisymmetric matrices, the Pfaffian is a square root of the determinant,

$$\det X = Pf(X)^2.$$

If X is real and anti-symmetric, we can block-diagonarize it by $SO(2r)$ as

$$X = \begin{pmatrix} 0 & x_1 & 0 & 0 & \cdots & 0 & 0 \\ -x_1 & 0 & 0 & 0 & \cdots & \cdot & 0 \\ 0 & 0 & 0 & x_2 & & & \\ 0 & 0 & -x_2 & 0 & & & \\ \cdot & \cdot & & & \cdot & & \\ 0 & \cdot & & & & 0 & x_r \\ 0 & 0 & & & & -x_r & 0 \end{pmatrix}$$

We can then write the Pfaffian as

$$Pf(X) = (-1)^r \prod_{i=1}^{r} x_i.$$

Under the conjugation $X \to \Omega^t X \Omega$, the Pfaffian transforms as

$$Pf(\Omega^t X \Omega) = \det\Omega \cdot Pf(X).$$

Thus, if $\Omega \in SO(2r)$, the Pfaffian is invariant. (Note that if $\Omega \in O(2r)$, the Pfaffian may change its sign.)

We can now define the Euler class by $e(F) = Pf(F)$.

In particular, the tangent bundle TM of an orientable Riemannian manifold M of dimensions $n = 2r$ is an $SO(2r)$ bundle. For example,

$$\begin{aligned} n = 2: \quad e(TM) &= \frac{1}{2\pi} R_{12}, \\ n = 4: \quad e(TM) &= \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} R^{\mu\nu} \wedge R^{\rho\sigma}, \end{aligned} \tag{8}$$

where the Riemann curvature is regarded as the 2-form as,

$$R^a{}_b = \frac{1}{2} R_{cd}{}^a{}_b e^c \wedge e^d = \frac{1}{2} R_{\mu\nu}{}^a{}_b dx^\mu \wedge dx^\nu.$$

The Gauss-Bonnet theorem for an even-dimensional manifold M relates the Euler characteristic $\chi(M) = \sum_p (-1)^p b_p$ to the Euler class by

$$\chi(M) = \int_M e(TM).$$

The Pontryagin classes are defined similarly to the Chern classes as

$$p(E) = 1 + p_1(E) + p_2(E) + \cdots = \det\left(1 - \frac{1}{2\pi}F\right).$$

Since F is anti-symmetric, we only have nontrivial polynomials with even degrees in F. Thus, we choose $p_i(E)$ to be a $4i$-form. The highest Pontryagin class is at $i = r$ where $2r$ is the dimension of the fiber (unless $2r > n = \mathrm{dim}M$). At this highest degree, it is the square of the Euler class,

$$p_r(E) = e(E)^2.$$