Lecture 6

topological space, homology

ϵ-δ definition

In the earlier lecture, we assume that we know when a function $f : M \to \mathbf{R}$ is continuous. To define a continuous function, we need to define a topological space. Here we will give such a definition.

Before we discuss a general notion of topological spaces, let us remind ourselves on the ϵ-δ definition of limit of a function $f : \mathbf{R} \to \mathbf{R}$. The formula,

$$\lim_{x \to x_0} f(x) = y,$$

means that "for any real $\epsilon > 0$, there is always $\delta > 0$ such that, for every x satisfying $|x - x_0| < 0$, we have $|f(x) - y| < \epsilon$."

This method is also used to define continuous functions (often attributed to Karl Weierstrass). We say that $f(x)$ for $x \in \mathbf{R}$ is continuous if, for any $x \in \mathbf{R}$ and $\epsilon > 0$, there is always $\delta > 0$ such that for every y satisfying $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$.

A closely related notion is a Cauchy sequence. An infinite sequence of numbers $\{x_1, x_2, ...\}$ is called Cauchy if and only if, "for any $\epsilon > 0$, there is always an integer $N > 0$ such that, for any pair of integers $n, m > N$ such that $|x_n - x_m| < \epsilon$." In the set of real numbers, every Cauchy sequence has a limit. This is not the case for the set of rational numbers. We say that the real numbers are complete, but the rational numbers are not.

One way to define the real numbers is to use a set of all possible Cauchy sequences in the rational numbers. We can add and multiply two Cauchy sequences. We say that two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ define the same real number if, for any $\epsilon > 0$ there is always an integer $N > 0$ such that $|x_n - y_n| < \epsilon$ for any $n > N$. The set of all Cauchy sequences with this equivalence relation defined the real numbers.

topological space

We note that the inequality symbol $>$ plays a prominent role in these definitions. In fact, we can define a continuous function on $\mathbf{R}$ by using open intevals $(a, b) = \{x : a < x < b\}$ alone. Thus, to define continuous functions on a general space M, we will introduce on M a generalization of the notion of open intervals. They are called open sets of M. To do so, we start with the following list of basic properties of open intervals of $\mathbf{R}$:

Let $\mathcal{U}$ be a set of unions of open intervals on $\mathbf{R}$. This means that

(0) $\mathcal{U}$ contains (a, b) $(a < b)$, and

(1) If $\{U_a \in \mathcal{A} : a \in S\}$ is a (possibly infite) subset of $\mathcal{U}$ parametrized by A, then $\cup_{a \in A} U_a \in \mathcal{U}$.

(2) We also assume $\phi, \mathbf{R} \in \mathcal{U}$. (Here, ϕ means the empty set.)

There is one more property of open intervals:

(3) If $U_1, U_2, ..., U_k$ (k: positive integer) are in $\mathcal{U}$, so is $\cap_{i=1}^{k} U_i$.

Question 1: Why do we consider only finite subsets of $\mathcal{U}$ in (3) while we allow infinite subsets in (2)?

To definte open sets for M, we generalize the above properties of open intervals of $\mathbf{R}$.

Definition of topological space: Suppose M is a set. It is called a topological space if there is a set $\mathcal{U}$ of subsets of M satisfying the following properties.

(1) $\phi, M \in \mathcal{U}$.

(2) If $\{U_a \in \mathcal{U} : a \in A\}$ is any subset of $\mathcal{U}$, then $\cup_{a\in A} U_a \in \mathcal{U}$.

(3) If f $U_1, U_2, ..., U_k$ are in $\mathcal{U}$, so is $\cap_{i=1}^{k} U_i$.

In this case, we call $U \in \mathcal{U}$ an open set of M. The set $\mathcal{U}$ of open sets *defines* the topology of M. For a point p, an open set $U \in \mathcal{U}$ containing p is called an open neighborhood of p.

removing pathological cases

The definition of topological spaces in the above is the best attempt by mathematicians to generalize the notion of continuity. However, this allows a few pathological cases. For example, for any set M, we can define a set $\mathcal{U} = \{\phi, M\}$. This satisfies all the properties (1)-(3) in the above. This is a funny topology. All the points in M are in the same neighborhood. To avoid such pathological cases, Hausdorff introduced an additional assumption. We say that a topological space M is Hausdorff iff, for any $p, q \in M$ $(p \neq q)$, we can find open neighborhood U, V $(p \in U, q \in V)$ such that $U \cap V = \phi$. This is to say that topology can separate two different points.

Often, we emploly one more assumption, the second countability assumption. It states that we can choose a countably many elements $U_{ii\in\mathbf{Z}}$ so that their unions and intersetions make all open sets in $\mathcal{U}$.

Question 1: Generalize the Weierstrass definition of continuous functions on $\mathbf{R}$

When we can choose $\{U_i \in \mathcal{U}\}_i$ so that there is a continuous one-to-one map $f_i : U_i \to \mathbf{R}^n$, we call M a topological manifold.

triangulation

A simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimension. A p-simplex is a p-dimensional polytope which is the convex hull of its $(p+1)$ vertices. A simplicial complex K is a topological space of a particular kind, constructed by gluing them together. For a topological space M, if we can find a topologically equivalent simplicial complex K (namely, when there is a continuous map $K \to M$ that is one-to-one and onto), we call it a triangulation of M. We can also consider a smooth (C^∞) triangulation. That is a triangulation so that, for each simplex in K, the map gives its C^∞ embedding in M. (C^∞ means that it is differentiable infinite times.)

In order to study geometric objects, it is often convenient to use their triangulations. It is known that every C^∞ manifolds has a smooth triangulation.

homologies

A simplicial complex K contains simplexes of dimensions $0, 1, ..., n$. Each p-simplex σ has $(p+1)$ vertices $v_0, ..., v_p$. An orientation of the simplex is determined by an ordring of the vetices, modulo even permutations. So let us denote such an oriented simplex by $\langle \sigma \rangle = \langle v_0, ..., v_p \rangle$. We can consider a vector space with integer coefficients (free Abelian group) whose basis vectors are p-simplexes of K, and denote it by $C_p(K)$. If σ' is the same simplex as σ but with the opposite orientation, we write $\langle \sigma' \rangle = -\langle \sigma \rangle$. An element of $C_p(M)$ is a sum of simplexes with integer coefficients, and we call them p-chais. We can formally add or subtract p-chains in $C_p(K)$.

The boundary operator ∂ is a map from $C_p(K)$ to $C_{p-1}(K)$. On each simplex, it acts as

$$\partial \langle v_0, ..., v_p \rangle = \sum_{i=0}^{p} (-1)^i \langle v_0, ..., \hat{v}_i, ..., v_p \rangle, \tag{1}$$

where $\hat{v}_i$ means we remove v_i in the right-hand side. For example, the boundary of a triangle $\langle v_0, v_1, v_2 \rangle$ is a sum of its 3 edges, $\langle v_1, v_2 \rangle, \langle v_2, v_0 \rangle, \langle v0, v_1 \rangle$. Similarly, the boundary of a tetrahedron $\langle v_0, v_1, v_2, v_3 \rangle$ is a sum of its four faces, $\langle v_1, v_2, v_3 \rangle, ...$ We can define the action of ∂ on $C_p(K)$ by extending this linearly.

Question 2: Prove that the boundary operator is nilpotent, $\partial^2 = 0$.

Given this, we can define homologies just as we defined the de Rham cohomology. We introduce sets of p-dimensional cycles Z_k and boundaries B_k as follows,.

$$Z_p(K) = \{c \in C_p(K) : \partial c = 0\} \quad B_p(K) = \{\partial c : c \in C_{p+1}(K)\}. \tag{2}$$

Cycles are chains without boundaries. Homologies are defined as their quotients,

$$H_p(K) = Z_p(K)/B_p(K). \tag{3}$$

When M is topologically equivalent to K, we define $H_p(M) = H_p(K)$. The homologies we defined in the above is with integer coefficients, so often denoted by $H_p(M, \mathbf{Z})$. We can also consider $H_p(M, \mathbf{R})$ and $H_p(M, \mathbf{C})$. There are simple relations such as $H_p(M, \mathbf{R}) = H_p(M, \mathbf{Z}) \otimes \mathbf{R}$. Thus, these homologies are essentially the same, except for finite groups possibily in $H_p(M, \mathbf{Z})$. These finite groups, if exist, are called torsions.

Question 3: Show that $H_0(M, \mathbf{R}) = \mathbf{R}$ for any connected manifold M.

For the torus T^2, which we can think of a surface of a doughnut, we have $H_1(T^2, \mathbf{R}) = \mathbf{R} \oplus \mathbf{R}$ and $H_2(T^2, \mathbf{R}) = \mathbf{R}$.

As an example of a torsion, we can consider the group $SO(3)$. As a manifold, $SO(3)$ is equivalent to S^3 with antipodal points identified. This can be used to show $H_1(SO(3), \mathbf{Z}) = \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$, while $H_1(SO(3), \mathbf{R}) = 0$.

simplicial cohomologies, de Rham cohomologies

Simplicial cohomologies are dual spaces of homologies $H_p(M, \mathbf{R})$ of M. The fundamental theorems of de Rham implies that it is isomorphic to the de Rham cohomologies $H^p(M)$. To explain the theorems, we note that there is a natural inner product between a p-cycle c in $Z_p(M)$ and closed p-forms ω in $Z^p(M)$ as follows.

$$(c, \omega) = \int_c \omega. \tag{4}$$

Suppose γ is a p-chain, but is not necessarily a cycle (∂c may not be zero). The Stokes theorem says,

$$\int_\gamma d\lambda = \int_{\partial\gamma} \lambda. \tag{5}$$

Therefore, if $c \in Z_p(M)$ and $\omega \in Z^p(M)$, then for any $(p+1)$-chain γ and any $(p-1)$-form λ,

$$\begin{aligned} (c, \omega + d\lambda) &= (c, \omega) + \int_{\partial c} \lambda = (c, \omega), \quad (\text{since } \partial c = 0) \\ (c + \partial\gamma, \omega) &= (c, \omega) + \int_\gamma d\omega = (c, \omega) \quad (\text{since } d\omega = 0). \end{aligned} \tag{6}$$

Therefore, (c, ω) is independent of the choice of representatives of the equivalence classes $H_p(M)$ and $H^p(M)$ and defines a pairing between them. The pairing (c, ω) is also called a period integral.

Now, we are ready to state the theorems of de Rham when M is a compact manifold without boundary. Let us choose a basis $\{c^i\}$ $(i = 1, ..., \mathrm{dim}H_p)$ of $H_p(M)$.

Theorem 1: For any set of integers ν_i, there is a closed p-form ω such that

$$\int_{c^i} \omega = \nu_i. \tag{7}$$

Theorem 2: If all the period of a p-form ω are zero,

$$\int_{c^i} \omega = 0, \tag{8}$$

then ω must be exact (*i.e.* $= 0$ in $H^p(M)$). This means that, if $\{\omega_i\}$ is a basis of $H^p(M)$, the period matrix π_{ij} defined,

$$\pi_{ij} = \int_{c^i} \omega_j, \tag{9}$$

is invertible.

In particular, $\mathrm{dim}H_p(M) = \mathrm{dim}H^p(M) = b_p$, the Betti numbers.

Poincaré duality

More generally, Poincaré duality states that $H^p(M)$ and $H^{n-p}(M)$ are dual to each.

Poincaré duality is easy to demonstrate if M is a Riemannian manifold and is compact, orientable, and without boundary. To show this, remember that representatives of $H^p(M)$ can be chosen as p-forms ω which satisfies

$$d\omega = 0, \quad \delta\omega = 0. \tag{10}$$

We have the Hodge dual operator $*$ which maps a p-form to an $(n-p)$-form. Since $\delta \sim *d*$ and since $** \sim 1$, we find that the Hodge operator maps $H^p(M)$ to $H^{n-p}(M)$ and vice versa. In particular, $H^n(M) = \mathbf{R}$ $(n = \mathrm{dim}M)$ since any $\omega \in H^n(M)$ can be written as being proportional to the volume form.