Lecture 5

vector bundles, gauge theory

tangent bundle

In Lecture 2, we defined the tangent space T_pM at each point p on M. Let us consider a collection of tangent bundles over every point on M,

$$TM = \cup_{p \in M} T_pM.$$

It is naturally a manifold. For a given coordinate chart (U_i, ϕ_i), we can define coordinates on $\cup_{p \in U_i} T_pM$ as (x^μ, v^μ), where (x^μ) are coordinates on U_i and we parametrize a tangent vector as

$$v = v^\mu \frac{\partial}{\partial x^\mu}.$$

This defines differential structure on TM (namely, TM is a differential manifold). TM is called a tangend bundle.

A smooth vector field is $v : p \in M \to v(p) \in T_pM$ such that its components v^μ expressed in coordinates x^μ are smooth functions of the coordinates on each U_i. We also call it a smooth *section* of TM. The reason for this name is as follows. The tangent bundle TM is locally a product space, $U_i \times \mathbf{R}^n$. Imagine that U_i is stretched in a horizontal direction and $\mathbf{R}^n$ in a vertical direction. The vector field v is then a graph over U_i, which lifts U_i in $U_i \times \mathbf{R}^n$. It cuts TM along the direction of M, which is why it is called a section.

When we change coordinates, $x^\mu \to \tilde{x}^\mu(x)$, the tangent space coordinates change as

$$v^\mu \to \tilde{v}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} v^\nu,$$

so that $v = v^\mu \partial/\partial x^\mu = \tilde{v}^\mu \partial/\partial \tilde{x}^\mu$ is independent of coordinates.

vector bundle

Vector bundles generalize the notion of the tangent bundle TM. On each coordinate chart $(U_i \phi_i)$, it should be of the form $U_i \times V$ for some vector space V. (dim V does not have to be the same as dim M.) To define a vector bundle more abstractly, mathematicians say that a differential manifold E is a vector bundle if

(1) there is a projection map π,

$$\pi : E \to M,$$

so that, for each point $p \in M$, its inverse image $\pi^{-1}(p)$ is isomorphic to V. For the tangent bundle TM, $\pi^{-1}(p) = T_pM$.

(2) we can choose atlases of E and M so that, for each local coordinate chart U of M, there is a smooth map $\varphi : \pi^{-1}(U) \to U \times V$. The map φ is called local trivialization of the vector bundle E over U.

The vector space V, which sits on the top of each $p \in M$, is called a fiber. When V is a vector space over $\mathbf{R}$, which we will consider in this lecture, we say that E is a real vector bundle. We can also consider a complex vector bundle. In particular, when V is 1-dimensional over $\mathbf{C}$, we say that E is a line bundle.

Suppose two coordinate charts U_i and U_j of M overlap with each other. Over $U_i \cap U_j$, there are two local trivializations φ_i and φ_j. Their composition, $\varphi_i \circ \varphi_j^{-1}$, maps $U_i \cap U_j \times V$ to itself as

$$\varphi_j \circ \varphi_i^{-1}(p, v) \to (p, g_{j \leftarrow i}(p) v),$$

where $p \in M$, $v \in V$ and $g(p)$ is an invertible linear map on V. This $g_{j\leftarrow i}(p) \in GL(V, \mathbf{R})$ is called a transition function.

If there is a triple intersection of three charts U_i, U_j and U_k, the transition function must satisfy the consistency condition,

$$g_{k\leftarrow j}(p) g_{j \leftarrow i}(p) = g_{k \leftarrow i}(p),$$

on $p \in U_i \cap U_j \cap U_k$. This is called a cocycle condition.

Conversely, if we have a differential manifold M, and if we have a transition function $g_{j\leftarrow i}(p) \in GL(V, \mathbf{R})$ for $p \in U_i \cap U_j$ satisfyin the cocycle condition, then there is a unique vector bundle E over M.

For the tangent bundle TM, we considered a tangent vector field v, which we may consider as a map $p \in M \to v(p) \in T_p$. Similarly, for a general vector bundle E, we may consider a map $s : p \in M \to s(p) \in \pi^{-1}(p)$. An example of s is the zero section where $s(p) = 0 \in V$ for all p.

fiber bundle

We can consider a more general class of manifolds E called fiber bundles, where there is a projection $\pi : E \to M$, but the fiber $F \sim \pi^{-1}(p)$ for $p \in M$ is not necessarily a vector space. For example, one can consider the case where the fiber is a group G. Over a coordinate chart U of the base manifold M, E looks like $U \times G$. When two charts U_i and U_j overlap the transition function is given by $(p, g \in G) \to (p, g(p)\rho \in G)$, where $g(p) \in G$.

When E is a vector bundle, we can consider its associated principal bundle whose transition functions are given by those of E.

example: magnetic mnopole bundle and Hopf fibration

Consider the 2-sphere S^2 and a $U(1)$ principal bundle E over S^2. As a manifold, the group $U(1)$ can be regarded as a circle S^1; the angle coordinate $\theta \in [0, 2\pi)$ of S^1 gives an element $e^{i\theta} \in U(1)$. Thus, we are considering an S^1 bundle over S^2.

As we discussed in Lecture 1, S^2 can be covered by 2 coordinate charts, U_+ and U_-. They can be chosen so that U_+ (U_-) contains the northen (southern) pole and that they overlap in a region near the equator of S^2. We can choose their coordinates as $(r_\pm, \phi)$, where $t_\pm$ is a distance from the northen (southen) pole and ϕ is the longitude of S^2.

We can then choose two coordinate charts of E. Over $U_\pm$, we can use $(r_\pm, \phi; \theta_\pm)$, where $(r_\pm, \phi)$ are coordinates of $U_\pm$ and $\theta_\pm$ parametrizes the S^1 fiber.

Let us consider the transition function,

$$e^{i\theta_-} = e^{in\phi} e^{i\theta_+},$$

for some integer n. This represents the configuration of the electro-magnetic field in the presence of a magnetic monopole of charge n.

When $n = 0$, the principal bundle is a trivial product, $E_{n=0} = S^2 \times S^1$.

When $n = 1$, the total space of the principal bundle makes the 3-sphere,

$$E_{n=1} = S^3.$$

This is known as the Hopf fibration. To exhibit the fibration structure, let us present S^3 as a subspace of $\mathbf{R}^4$ subject to the condition,

$$a^2 + b^2 + c^2 + d^2 = 1.$$

This is to be identified with the total space of the bundle E. This bundle is suppose to have S^2 has a base manifold, so we need to exhibit the projection map $\pi : S^3 \to S^2$. This, according to Heiz Hopf, is given by

$$x = a^2 + b^2 - c^2 - d^2, \; y = 2(ad + bc), \; z = 2(bd - ac).$$

It is elementary to verify that $z^2 + y^2 + z^2 = 1$.

Question 1: Show that $\pi^{-1}(p) \sim S^1$ for each $p \in S^2$.

(Hint: Introducing $u = a + ib$ and $v = c - id$, we can express the equation for S^3 as

$$u\bar{u} + v\bar{v} = 1.$$

The projection map $\pi : S^3 \to S^2$ is

$$x = u\bar{u} - v\bar{v}, \quad z + iy = 2u\bar{v}.$$

If we fix (x, y, z), what are the remaining degrees of freedom on S^3?)

connection and curvature

As we discuss in Lecture 3, the problem with defining partial derivatives of a tangent vector field on M is that, a priori, there is no indentification of T_pM and $T_{p'}M$ even when p and $p' \in M$ are closed to each other. To define a derivative, we need a way to perform *paralell transport* of a vector v along a smooth path $c(t)$ on M. Consider a smooth path $c(t)$ and an arbitrary vector $u \in T_pM$ at $p = c(t = 0)$. A parallel transport means that we can define $\Omega(t) \cdot u \in T_{c(t)}M$. Since the tangent space is a linear space, we are writing the parallel transport as a linear map $u \to \Omega(t) \cdot u$. Then, we can define a covariant derivative ∇_t of a vector field $v(x)$ at $p \in M$ as

$$\nabla_t v = \frac{d}{dt}\big[v(x(t)) - \Omega(t) \cdot v(x(t = 0))\big].$$

Since we can choose $c(t)$ to be tangent to any direction at p, this defines a covariant derivative. (For example, if we want to compute $\nabla_i v$ in the x^i direction, we can just choose $c(t)$ to be $(x^1, ..., x^i + t, ..., x^n)$.

For a Riemannian manifold, $\Omega(t)$ is uniquely determined by requiring that the covariant derivative of the metric, which is a section of $T^*M \otimes T^*M$, is zero, and that the torsion tensor of the connection is zero.

This can be done for any vector bundle. For each point $p \in M$, there is a vector space $\pi^{-1}(p)$. To define a covariant derivative, we introduce a parallel transport, which is a linear map $\Omega(t) : \pi^{-1}(c(t = 0)) \to \pi^{-1}(c(t))$ along any smooth curve $c(t)$. In fact, all we need is an infinitesimal limit of this since we just need to take one derivative with respect to t. The infinitesimal version of the parallen transport should give a one-form, valued in matrix on V, where V is the fiber over p, since it should give a linear map on $V = \pi^{-1}(p)$ to any direction along the tangent space T_pM. This one-form is called a *connection form.*

Pick a coordinate chart $(U_i, \mathbf{R}^n)$ of a vector bundle E. The covariant derivative of a section of E, expressed in the coordinates as $(v(x)^\alpha)_{\alpha=1,\ldots,n}$, can be written as

$$\nabla_\mu v^\alpha(x) = \partial_\mu v^\alpha(x) + A_{\mu\ \beta}^{\ \alpha} v^\beta(x),$$

where $A^\alpha_{\ \beta} = A_{\mu\ \beta}^{\ \alpha} dx^\mu$ is a matrix valued connection of E.

When two coordinate charts U_i and U_j overlap, coordinates on the fiber over U_i and over U_j are related by a linear map as $v \to g(p)v$ for some $g \in GL(V, \mathbf{R})$. To be compatible with the derivative operation, the connection form should transform as

$$A \to g^{-1}Ag + g^{-1}dg.$$

Question 2: Show that the spin connection $\omega_{\mu\ b}^{\ a}$ defined in Lecture 3 transforms as a connection.

The curvature F for the connection is a matrix-valued 2-form defined by

$$F = dA + A \wedge A.$$

In components, one can show that

$$F_{\mu\nu} v(p) = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, v(p),$$

for any smooth section $v(p)$ of E. Under the change of coordinates of the fiber, $v \to g(p)v$, the connection 2-form gtransforms as

$$F \to g^{-1}Fg.$$

holonomy

Pick any point $p \in M$ and move around M along a closed path γ and come back to the same point p. We can parallel transport a vector v in the fiber $\pi^{-1}(p)$ along the path. When we come back to p, the vector v is rotated to $g(\gamma)v$ by some element $g(\gamma) \in GL(V, \mathbf{R})$. It is called a holonomy along γ. If we have two such paths γ_1 and γ_2, we can combine them (start at p, go around γ_1 to come back to p, then start at p again and go around γ_2) to make another path γ_3. It is easy to show that $g(\gamma_3) = g(\gamma_2)g(\gamma_1)$. Thus holonomies along closed paths starting and ending at p makes a subgroup of $GL(V, \mathbf{R})$. It is called a holonomy group.

Question 3: Suppose any two points on M can be connected by a path on M. Show that holonomy groups at two different points p and q are isomorphic. (Two groups G_1 and G_2 are called isomophic if there is a map $f : G_1 \to G_2$ that is one-to-one and onto and if the map respect the group operations, $f(gg') = f(g)f(g')$ for $g, g' \in G_1$.

The curvature $F_{\mu\nu}$ is a holonomy for an infinitesimal loop.

When the curvature vanishes, the holonomy for a look γ is invariant under continuous deformation of γ. In that case the holonomy depends only on topological (global) data of γ. It is called monodoromy in that case.

The holonomy group of an n-dimensional Riemannian manifold M is a subgroup of $SO(n)$. If M is a Kähler manifold and $n = 2m$, its holonomy group is a subgroup of $U(m)$. If M is a Calabi-Yau manifold, its holonomy group is a subgroup of $SU(m)$.

gauge theory

Consider a vector bundle with a complex 1-dimensional fiber. A section s is a complex-valued function in each coordinate patch and transforms as $s(p) \to g(p)s(p)$ under a change of coordinates, where $g(p)$ is a non-vanishing complex-valued function. The covariant derivative takes the form, $\nabla_\mu s(p) = (\partial_\mu + iA_\mu)s(p)$ for a complex-valued connection form A_μ. (I included the imaginary unit in front of A_μ for a later convenience.) Under the change of coordinate, the connection transforms as

$$A \to A - id \log g(p).$$

Since everything commutes over complex number, $g^{-1}A_\mu g = A_\mu$.

If we restrict the transition function $g(p)$ to be in $U(1)$ and write $g(p) = e^{i\lambda(p)}$ for some real valued function λ,

$$A_\mu \to A_\mu + \partial_\mu \lambda,$$

and the curvature 2-form $F = dA$ is given in components by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

If we identify A_μ as the vector potential of the Maxwell theory of electro-magnetism, these are the gauge transformation rule and the definition of the field strength.

For a vector bundle with a higher dimensional fiber, the connection form A_μ is matrix-valued, and the curvature is given in components by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu],$$

where I made the substitution $A \to iA$, in comparison with the convention in the previous section. This gives a non-Abelian generalization of the Maxwell theory, known as the Yang-Mills theory.