Lecture 3

Cohomologies, curvatures

Maxwell equations

The Maxwell equations for electromagnetic fields are expressed as

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{H}}{\partial t}, \quad \vec{\nabla} \cdot \vec{H} = 0,$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \quad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{E}}{\partial t} = 4\pi\vec{j}.$$

These equations can be simplified if we use the 4-dimensional notation as

$$\partial_\mu F_{\nu\rho} - \partial_\nu F_{\mu\rho} = 0, \quad \partial^\mu F_{\mu\nu} = 4\pi J_\nu,$$

where an anti-symmetric tensor field $F_{\mu\nu}$ and a vecfor field J_μ are given by by

$$F_{0i} = E_i, \quad F_{ij} = -\epsilon_{ijk} H_k,$$

$$J_0 = \rho, \quad J_i = -j_i.$$

It is natural to regard them as differential forms, $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ and $J = J_\mu dx^\mu$. The Maxwell equations can then be expressed as

$$dF = 0, \quad \delta F = -4\pi J.$$

The equation $dF = 0$ can be solved if we set $F = dA$. On $\mathbf{R}^4$, one can show conversely that any solution to $dF = 0$ is given by $F = dA$ for some A. This is an example of the Poincaré lemmna. In the terminology of electromagnetism, $\phi = A_0$ is the scalar potential and $-A_i$ is the vector potential.

The choice of A is not unique. If $F = dA$, we can replace A by $A' = A + d\lambda$ and we still have $F = dA'$. The Maxwell equations have gauge symmetry.

Question 1: We can restrict the gauge degrees of freedom by imposing the condition $\delta A = 0$. For any F, can we choose an appropriate gauge to satisfy this condition? Is there any remaining gauge symmetry after imposing this condition? Discuss this for both a positive definite metric and a Lorentzian signature metrics.

co-homologies

We can generalize the relation between the field strength 2-form F and the gauge potential 1-form A to other differential forms. If a k-form ω satisfies $d\omega = 0$, it is called closed. If it can be written as $\omega = d\lambda$ for some $(k-1)$-form λ, we say ω is exact. Clearly, if an exact form is closed. On $\mathbf{R}^n$, the converse is also true.

[Poincaré Lenma]

On $\mathbf{R}^n$, if a k-form ω is closed, it is also exact.

This also holds when M is contractible, namley when one can smoothly shrink M to a point. This is not true on a general manifold. However, it is true on each coordinate chart. The issue is how these charts are patched together globally.

It is interesting to ask what are closed differential forms that are not exact. This leads us to introduce the concept of cohomology. We first consider a space of closed k-forms,

$$Z^k(M) = \{\omega \in C^k(M) : \ d\omega = 0\}.$$

Clearly, if ω is closed, so is $\omega + d\lambda$. So, we can introduce an equivalence relation in Z^k: $\omega \sim \omega'$ if and only if their difference is exact. The k-th de Rham cohomology $H^k(M)$ is defined as a quotient of Z^k by the space of exact forms,

$$B^k(M) = \{d\lambda : \ \lambda \in C^{k-1}(M)\},$$

as

$$H^k(M) = Z^k(M)/B^k(M).$$

The dimensions of H^k is called the Betti number,

$$b_k = \mathrm{dim} H^k(M).$$

The Betti numbers are topological invariant of M. In particular, their alternative sum is the Euler characteristic,

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b_k.$$

Question 2: Show that $b_0 = 1$ if M is connected. What if M has m disjoint components?

Question 3: Compute the first cohomology H^1 of a circle S^1.

$H^k(M)$ is called co-homology since it is related to an object called homology. That will be discussed later.

representatives of cohomologies

If M is endowed with a Riemannian (positive definite) metric g, we can define the codifferential δ and a positive definite inner product (α, β) as discussed in Lecture 2.

Question 4: Express the inner product (α, β) of k-forms in their components.

It is easy to see that the Laplace-Beltrami operaotr $\Delta = \delta d + d\delta$ commutes with d. Thus, we can choose a solution to $d\omega = 0$ to be an eigenstate of Δ.

$$(d\delta + \delta d)\omega = \epsilon\omega.$$

If $\epsilon \neq 0$, since $d\omega = 0$, we have $\omega = d(\delta\omega/\epsilon)$, which would mean that ω is exact. So, if we want a closed but not exact form, it should be a zero eigenstate of Δ, i.e., a harmonic form. Thus, we can choose $\omega \in H^k(M)$ as a harmonic form.

In fact, this fixes the gauge degrees of freedom. To see that, we write

$$(\omega, \Delta\omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega).$$

and observe that the right-hand side is a sum of non-negative numbers. Therefore, ω is harmonic if and only if ω is closed ($d\omega = 0$) and co-closed ($\delta\omega = 0$). Now, suppose ω and $\omega + d\lambda$ are both harmonic. This means in particular that $\delta d\lambda = 0$. However, $(d\lambda, d\lambda) = (\lambda, \delta d\lambda) = 0$. Thus, $d\lambda = 0$. Namely, if we require ω to be harmonic, there is no more gauge degrees of freedom. Since $\Delta\omega = 0$ is equivalent to $d\omega = 0$ and $\delta\omega = 0$, this means that we can choose representatives ω of $H^k(M)$ to be co-closed $\delta\omega = 0$.

The condition $\delta\omega = 0$ is a generalization of the Lorentz gauge condition in the Maxwell theory.

It is also useful to know that any k-form on a compact orientable Riemannian manifold can be uniquely expressed as a sum of an exact form, a co-exact form, and a harmonic form. This is called the Hodge decomposition.

As we saw in the last lecture, the operators d and δ are analogous to the supersymmetry operators (supercharges) acting on the space of differential forms $\oplus_k C^k(M)$. Differential forms with k even are bosons and those with k odd are fermions. The supercharges map bosons to fermions and vice versa. The Laplace-Beltrami operator $\Delta = \{d, \delta\}$ is analogous to the Hamiltonian. If ω is a supersymmetric state, $d\omega = 0, \delta\omega = 0$. This means that $\Delta\omega = 0$. Namely, any supersymmetric state has zero energy. Some people thinks that this may help us explain why the dark energy (cosmological constant) of the Universe is so small.

covariant derivatives, curvature

The exterior derivative is a special derivative operator defined for differential forms. It is also just one derivative operator. On the other hand, to derive differential equations on M, we may want more differential operators. Unfortunately, on a general differnetiable manifold M without futhre structure, it is not possible to define a partial derivative $\partial/\partial x^i$ on a general tensor field, as it depends on a choice of coordinates.

Another say to point out the problem is as follows. If we have a smooth function $f : M \to \mathbf{R}$, we can define its partial derivatives with respect to coordinates x^i as

$$\frac{\partial f}{\partial x^i} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(f(x^1, ..., x^i + \epsilon, ..., x^n) - f(x^1, ..., x^i, ..., x^n) \right).$$

This requires comparing values of f at x and $x + \epsilon$. Since f is taking values in $\mathbf{R}$, there is no problem with taking the difference of the two numbers, at x and $x + \epsilon$. This does not work for a tangent vector v, for example. At x it takes value in $T_x M$. At $x + \epsilon$, it takes value in $T_{x+\epsilon} M$. A priori, there is not natural identification of the two vector spaces, so we cannot compare the values of v at the two points.

The situation is better if we have a metric g_{ij}. With it, we can define the affine connection (Christoffel symbol) as,

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}).$$

A covariant derivative ∇_i of a rank-p tensor field $T_{i_1 \cdots i_p}$ can be defined as

$$\nabla_i T_{i_1 \cdots i_p} = \partial_i T_{i_1 \cdots i_p} - \Gamma^j_{ii_1} T_{j i_2 \cdots i_p} - \cdots - \Gamma^j_{ii_p} T_{i_1 \cdots i_{p-1} j}.$$

A covariant derivative of a tensor with upper indices, e.g., T^{ij}, can be defined similarly, with the $+$ sign instead of the $-$ sign in front of the affine connection Γ. In a later lecture, we will discuss why this procedure resolves the issue of comparing T_xM and $T_{x+\epsilon}M$. For now, I just point out that, if T transforms as a tensor of rank (r, s), then ∇T transforms as a tensor of rank $(r+1, s)$.

The affine connection is defined in such a way that the metric is covariantly constant,

$$\nabla_i g_{jk} = 0,$$

and the torsion tensor defined as an antisymmetric part of the Christoffel symbol is zero,

$$T^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) = 0.$$

The Riemann curvatuve $R^i{}_{jkl}$ is defined as a failure of the covariant derivatives to commute with each other,

$$[\nabla_i, \nabla_j]T_k = R_{ijk}{}^l T_l.$$

Explicitly,

$$R_{ijk}{}^l = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ik}\Gamma^l{}_{mj} - \Gamma^m_{jk}\Gamma^l{}_{mi}.$$

The Riemann curvature satisfies the following properties,

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}.$$

The first half of the equation follows simply by the definition and the second half is a consequance of the fact that the metric is covariantly constant.

$$R_{[ijk]l} = 0,$$

where $[ijk]$ means the antisymmetrization of the 3 indices. This can be proven by writing $d^2\omega = 0$ in terms of the covariant derivatives.

$$\nabla_{[i} R_{jk]lm} = 0.$$

The last equation is called the Bianchi identity. It is an important identity, but is not easy to verify this directly using the component notation.

curvature as seen by differential forms

The above expressions are simplified if we use the language of differential forms. First remember the vielbein e^a_i satisfying $g_{ij} = \sum_a e^a_i e^a_j$. Use this to devine a basis of $C^1(M)$ as,

$$e^a = e^a_i dx^i.$$

The affine spin connection $\omega^a{}_b$ is a 1-form defined by

$$(1) \quad de^a + \omega^a{}_b \wedge e^b = 0.$$

(The left-hand side is the torsion 2-form. We are setting the torsion to be zero.)

The curvature 2-form is defined by

$$(1) \quad R^c{}_d = \frac{1}{2} R_{ab}{}^c{}_d e^a \wedge e^b = d\omega^c{}_d + \omega^c{}_f \wedge \omega^f{}_d.$$

Question 5: Show that the curvature defined in this way is related to the Riemann curvature discussed earlier by

$$R_{ij}{}^{k}{}_{l} = R_{ab}{}^{c}{}_{d} e_i^a e_j^b e_c^k e_l^d.$$

By taking the exterior derivative of (1), we find

$$R^a{}_b \wedge e^b = 0,$$

which in components is equivalent to $R_{[ijk]l} = 0$.

Question 6: Show that the exterior derivative of (2) gives,

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = DR^a{}_b = 0,$$

and that it is equivalent to the Bianchi identity.