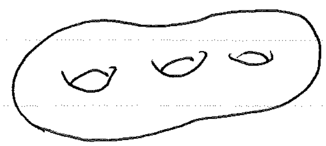
Topological String Theory

- string theory perturbative amplitudes

Start with 2d conformal field theory (CFT)
(↳ for Caltech students, more in the next week.)

2d surface Σ



with a metric
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

CFT : local scale invariance $g_{\mu\nu} \to \Omega g_{\mu\nu}$

$\Updownarrow$

$T^\mu{}_\mu$ = c-number

scalar curvature

locality + scaling $\Rightarrow$ $T^\mu{}_\mu = c R$

c : central charge.

In this case, CFT amplitudes depend only on conformal equivalence class of $g_{\mu\nu}$.

$$\{ g_{\mu\nu} \} / _{g_{\mu\nu} \sim \Omega g_{\mu\nu}} = \mathcal{M}$$: moduli space of complex structure of Σ.

g-loop string amplitudes

$$= \int_{\mathcal{M}} (\text{CFT amplitudes on } \Sigma \text{ with genus } g)$$

$$(\dim \mathcal{M} = 6g-6)$$

This makes sense only when $\underline{c=0}$.
(There are other conditions to define the measure.)

- sigma-model --- basic example of CFT

M : n-dim Riemannian manifold., G_{IJ} ; metric

sigma-model variable X^I ; $\Sigma \to M$

$$\mathcal{L} = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\, G_{IJ}(X)\, \partial_\mu X^I \partial_\nu X^J$$

metric on Σ metric on M

If we choose complex coordinate z on Σ,

$$ds^2 = 2\, g_{z\bar{z}}\, dz\, d\bar{z}$$

$$\mathcal{L} = \frac{1}{2} G_{IJ}(X)\, \partial_z X^I \partial_{\bar{z}} X^J$$

$\mathcal{L}$: $(1,1)$-form on Σ.

$S = \int_\Sigma \mathcal{L}$ is (classically) scale invariant.

There is another scale invariant term one can add.

$$B_{IJ}\, dX^I \wedge dX^J \in \Omega^2(M) \quad \text{(antisym 2 tensor)}$$

$$\epsilon^{\mu\nu} B_{IJ}(X)\, \partial_\mu X^I \partial_\nu X^J$$

$$= i\, B_{IJ}\, \partial_z X^I \partial_{\bar{z}} X^J$$

$$\mathcal{L} = \frac{1}{2} \underbrace{(G_{IJ} + i B_{IJ})}_{\text{complexification of the metric.}} \partial_z X^I \partial_{\bar{z}} X^J$$

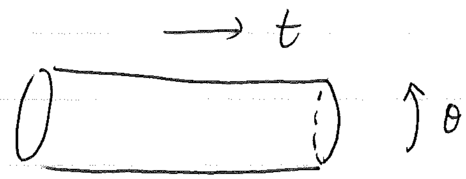
Question 1: Show that, when B_{IJ} is shifted by $\partial_I \Lambda_J - \partial_J \Lambda_I$, $\mathcal{L}$ changes by a total derivative on Σ.

- sigma-model on a torus

<u>Consider $M = S^1$</u>, radius R.

$$G_{11} = R^2, \quad B_{11} = 0$$

Consider a Lorentzian metric $(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ on Σ



cylindrical worldsheet

$$ds^2 = -dt^2 + d\theta^2$$

$$S = \int_0^{2\pi} d\theta \, \frac{R^2}{4\pi} \left((\partial_t X)^2 - (\partial_\theta X)^2 \right)$$

$$0 \leq X(t,\theta) \leq 2\pi$$

$X(t,\theta)$ represents ∞-many degrees of freedom

momentum conjugate to X : $P = R^2 \partial_t X$

$$H = \int_0^{2\pi} \frac{d\theta}{4\pi} \left(\frac{1}{R^2} P^2 + R^2 (\partial_\theta X)^2 \right)$$

- Since X is periodic, the center of mass momentum is quantized.

$$P = n + \text{oscillators}$$

(non-zero Fourier modes)

- As θ goes from 0 to 2π, X can wind S^1 several times.

$$X = m\theta + \text{oscillators}$$

$$n, m \in \mathbb{Z}$$

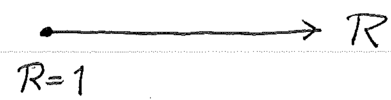
$$H = \frac{1}{2}\left(\left(\frac{n}{R}\right)^2 + (mR)^2 \right) + \begin{array}{l}\text{harmonic}\\ \text{oscillators}\end{array}$$

n : momentum , m : winding number

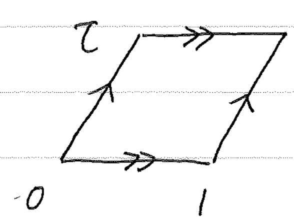
This is invariant under $R \to \frac{1}{R}$.

In fact, this is symmetry of this CFT
--- T-duality

Moduli space of S^1 :



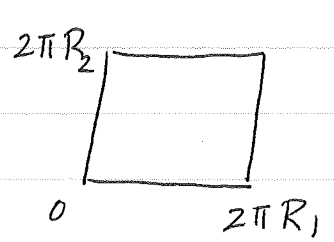
<u>2d torus T^2</u>



τ : complex structure moduli

$t = i\,(\text{area of } T^2) + B_{12}$
Kähler moduli.

For simplicity, consider · τ = pure imaginary
· $B_{12} = 0$



$\tau = i\, R_2/R_1$
$t = i\, R_1 R_2$

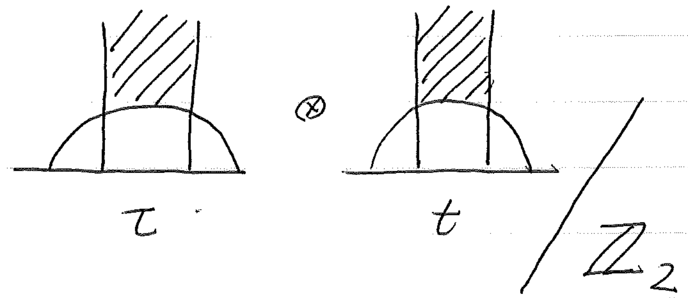
$R_1 \to 1/R_1 \iff t \to \tau$
$\tau \to t$

This holds even if τ, t : complex

$\tau \leftrightarrow t$: mirror symmetry

Since the moduli space of τ is $H/SL(2,\mathbb{Z})$, so is that of t.

Moduli space of T^2



- How about curved target space?

Quantum effects can break conformal invariance because of UV divergences.

e.g. QED with massless electron(s) is classically scale invariant, but the renormalization introduces a scale.

2d sigma-model with metric G_{IJ} is one-loop scale invariant if $R_{IJ} = 0$. (more general condition for $B_{IJ} \neq 0$)

Asymptotic free if $R_{IJ} > 0$.

- supersymmetric sigma-model.

c.f. supersymmetric quantum mechanics

$$X^I : \mathbb{R} \to M$$
$$\psi, \bar{\psi}^I : \mathbb{R} \to T_{X(t)} M$$

$\Rightarrow$ differential forms, superchanges $= d, d^+$.

Generalize this to 2d.

$$\Omega^{(m,n)}(\Sigma) \ni \omega (dz)^m (d\bar{z})^n$$

spinors : $\Omega^{(\frac{1}{2},0)}$, $\Omega^{(0,\frac{1}{2})}$ (Weyl basis)

$$\begin{cases} \psi_L{}^I : \Sigma \to T_X M \otimes \Omega^{(\frac{1}{2},0)} \\ \psi_R{}^I : \Sigma \to T_X M \otimes \Omega^{(0,\frac{1}{2})} \end{cases}$$

$$\mathcal{L} = \frac{1}{2} G_{IJ} \partial_z X^I \partial_{\bar{z}} X^J + \frac{i}{2} G_{IJ} \bar{\psi}^I \gamma^\mu D_\mu \psi^J \sqrt{g}$$
$$+ \frac{1}{12} R_{IJKL} \bar{\psi}^I \psi^J \bar{\psi}^K \psi^l$$

$$D_\mu \psi^I = \partial_\mu \psi^I + \Gamma^I{}_{Jk} \partial_\mu X^J \psi^k$$

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi$$
$$\psi_R = \frac{1}{2}(1+\gamma^5)\psi$$

Theorem: If M is a Calabi-Yau manifold,
i.e.

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K, \quad R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log \det G = 0,$$

then $G_{i\bar{j}}$ is invariant under renormalization
modulo $K \to K +$ globally defined function

In particular, complex structure and Kähler class
are not renormalized.

Namely, the supersymmetric sigma-model can be
made a CFT by appropriately adjusting K.

$$\begin{cases} X : \Sigma \to M \\ \psi_L^{i} : \Sigma \to T_X^{1,0} M \otimes \Omega^{(\frac{1}{2},0)} \\ \psi_L^{\bar{i}} : \Sigma \to T_X^{0,1} M \otimes \Omega^{(\frac{1}{2},0)} \\ \psi_R^{i},\ \psi_R^{\bar{i}} \end{cases}$$

can be changed to

so that
$g_{i\bar{j}} \psi^{\bar{j}} \bar{\partial} \psi^{i}$
is (1.1) form

$$\begin{cases} \otimes\, \Omega^{(p,0)} \\ \otimes\, \Omega^{(1-p,0)} \end{cases}$$

The central charges $\quad c_L = 12\, p_L (1-p_L)$

$\qquad\qquad\qquad\qquad\quad c_R = 12\, p_R (1-p_R)$

2 interesting cases :

A-model : $\quad \psi_L^{\,i} : \Omega^{(0,0)}, \quad \psi_L^{\,\bar{i}} : \Omega^{(1,0)}$

$\qquad\qquad\quad \psi_R^{\,i} : \Omega^{(0,1)}, \quad \psi_R^{\,\bar{i}} : \Omega^{(0,0)}$

B-model : $\quad \psi_L^{\,i} : \Omega^{(1,0)}, \quad \psi_L^{\,\bar{i}} : \Omega^{(0,0)}$

$\qquad\qquad\quad \psi_R^{\,i} : \Omega^{(0,1)}, \quad \psi_R^{\,\bar{i}} : \Omega^{(0,0)}$

In each model, there are 2 supersymmetries with scalar parameters $(\varepsilon, \bar{\varepsilon})$.

A-model
$$\delta X^i = \varepsilon\, \psi_L^{\,i} \qquad \delta X^{\bar{i}} = \bar{\varepsilon}\, \psi_R^{\,\bar{i}}$$
$$\delta \psi_L^{\,\bar{i}} = \varepsilon\, \partial X^{\bar{i}} \qquad \delta \psi_R^{\,i} = \bar{\varepsilon}\, \bar{\partial} X^i$$

SUSY configuration : $\bar{\partial} X^i = 0$

--- holomorphic map.

B-model
$$\delta X^i = 0$$
$$\delta X^{\bar{i}} = \varepsilon\, \psi_L^{\,\bar{i}} + \bar{\varepsilon}\, \psi_R^{\,\bar{i}}$$

$$\delta \psi_L^{\,i} = \varepsilon\, \partial X^i, \quad \delta \psi_R^{\,i} = \bar{\varepsilon}\, \bar{\partial} X^i$$

SUSY configuration : $\partial X^i = 0, \quad \bar{\partial} X^i = 0$

--- constant map

In A-model, supersymmetric amplitudes depend only on Kähler moduli.

In fact, $k \equiv (i G_{i\bar{j}} + B_{i\bar{j}}) dX^i \wedge dX^{\bar{j}}$

$$S \Big|_{\substack{\bar{\partial} X^i = 0 \\ \psi_- = 0}} = \int_\Sigma X^* k$$

In B-model, supersymmetric amplitudes depend only on complex structure.

$M, \tilde{M}$: mirror pair

$\Leftrightarrow$ A-model on M = B-model on $\tilde{M}$

e.g.

$T^2(\tau, t)$ and $T^2(t, \tau)$ make a mirror pair.