Lecture 10

homotopy

Consider continuous maps from a topological space X to another topological space Y. Two such maps are called homotopic if one can continuously deform one to another. This provides a useful way to define topological invariants. In particular, when X is the n-sphere S^n, the space of maps (modulo homotopy equivalence) becomes a group — the homotopy group — of Y, and the group is denoted by $\pi_n(Y)$.

Homotopy group should be contrasted with homology groups we studied earlier. For one thing, homotopy groups can be non-abelian, while homology groups are always abelian. In fact, $H_1(Y)$ is an abelian reduction of $\pi_1(Y)$. Thus, $\pi_1(Y)$ can contain more informaion on Y than $H_1(Y)$ does.

Fundamental Group

Pick a point x_0 in a topological space M. A continous map $\alpha : [0,1] \to M$ such that $\alpha(0) = \alpha(1) = x_0$ is called a loop or a closed path with x_0 as its base point. Suppose there are two such loops, α and β. They are called homotope, $\alpha \sim \beta$ if there is a continuous map $H : [0,1] \times [0,1] \to M$ such that,

$$\begin{aligned} H(t,0) = \alpha(t), \quad H(t,1) = \beta(t) \\ H(0,s) = H(0,t) = x_0 \end{aligned} \tag{1}$$

Namely, for each $s \in [0,1]$, $H(t,s)$ defines a loop with x_0 as its base point. As we vary s, H interpolates between α and β continuously.

Question 1: Show that $\alpha \sim \beta$ defines an equivalence relation, namely (1) $\alpha \sim \alpha$, (2) $\alpha \sim \beta$ implies $\beta \sim \alpha$, (3) $\alpha \sim \beta$, $\beta \sim \gamma$ implies $\alpha \sim \beta$.

The set of homotopy classes $[\alpha]$ of loops at x_0 is denoted by $\pi_1(M, x_0)$. It naturally is a group since we can combine two loops α and β to make another loop $\alpha * \beta$. The class containing the constant map $c : [0,1] \to x_0$ gives the identity. The group is called the fundamental group or the first homotopy group. For example, if we view S^1 as $\mathbf{R}/\mathbf{Z}$, we can choose representatives of the fundamental group as

$$\alpha_n : \theta \in [0,1] \to x_0 + n\theta \in \mathbf{R}.$$

Question 2: Show that $[\alpha_n] * [\alpha_m] = [\alpha_{n+m}]$ and that $\pi_1(S^1, x_0) = \mathbf{Z}$.

On the other hand, $\pi_1(S^2, x_0)$ is trivial since any loop on S^2 can be contractible to a point.

A topological space is called arcwise connected if any pair of points x_0 and y_0 in M can be connected by a continuous path, namely there is a continuous map $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$ and $\gamma(1) = y_0$.

Question 3: Show that $\pi_1(M, x_0) \simeq \pi(M, y_0)$ for any x_0, y_0 in M if M is archwise connected.

When M is archwisely connected, we can denote the fundamental group as $\pi_1(M)$ without refering to the base point.

Free Group and Relations

Homology groups are always abelian, but the fundamental group can be non-abelian.

It is a good place to introduce the notion of free group. Start with a finite set of letters, $X = \{a, b, c, ...\}$. Words are ordered lists of letters of the form, $\omega = x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}$ with $n_1, ..., n_N \in \mathbf{Z}$ and $x_1, ..., x_N \in X$. If we have $x^n x^m$, we can replace it with x^{n+m}. For example, $a^{-2}bc^3c^{-2}ae = a^{-2}bcae$. This is called reduction. In a reduced word, $x_i \neq x_{i+1}$ and $n_1, ..., n_N \neq 0$. The emply word gives the identity. We can take a produce of words, buy multiplying them together and then reducing the result. This defines the free group $F[X]$ finitely generated by X.

Consider the space $M = \mathbf{R}^2 - \{z_1, z_2\}$. The fundamental group $\pi_1(M)$ has two generators, one going around z_1, call a, and another going around z_2, call b. In fact, the fundamental group is a free group generated by a and b. Note that $aba^{-1}b^{-1}$ is not trivial. In general, $\pi_1(\mathbf{R}^2 - \{z_1, ..., z_N\})$ is a free group generated by n letters.

We can impose relations on a free group to construct another group. For example, $\mathbf{Z}_N = \mathbf{Z}/n\mathbf{Z}$ is x (regard $\mathbf{Z} = \{1, x^{\pm 1}, x^{\pm 2}, ...\}$) with the relation $x^N = 1$. The fundamental group of T^2 is generated by a and b with the relation $aba^{-1}b^{-1} = 1$. Thus, $\pi_1(T^2) = \mathbf{Z} \oplus \mathbf{Z}$.

Suppose G is an arbitrary finite group generated by $a_1, ..., a_N$. Consider the free group with the same number of generators, $F = F[\{A_1, ..., A_N\}]$. We can define a homomorphism, $\varphi : F \to G$ by $\varphi(A_{i_1}^{n_1} \cdots A_{i_N}^{n_N}) = a_{i_1}^{n_1} \cdots a_{i_N}^{n_N}$. By construction, G is surjective, $G = \text{Im}\varphi$. By the homomorphism theorem,

$$G = \text{Im}\varphi = F/\text{ker}\varphi.$$

We see that $\text{ker}\varphi$ gives a set of relations to define G from the free group F. This shows that any finitely generated discrete group can be constructed as a free group with relations. In particular, the fundamental group can also been seen in this way.

The first homology group of a genus-g Riemann surface Σ_g is $H_1(\Sigma_g; \mathbf{Z}) = \mathbf{Z}^{\oplus 2g}$. On the other hand, the fundamental group $\pi_1(\Sigma_g)$ is isomorphic to the quotient group of the free group on the generators $a_1, b_1, a_2, b_2, ..., a_g, b_g$ by the normal subgroup generated by $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. Except for $\pi_1(\Sigma_{g=0}) = 0$ and $\pi_1(\Sigma_{g=1}) = \mathbf{Z} \oplus \mathbf{Z}$, the group is non-abelian.

For a group G, its commutator subgroup F is generated by elements of the form $aba^{-1}b^{-1}$. It is a normal subgroup of G and the quotient group G/F is abelian. In fact, F is the smallest normal subgroup of G such that the quotient is abelian. One can show that the first homology group $H_1(M)$ is isomorphic to the fundamental group $\pi_1(M)$ divided by its commutator subgroup.

As an aside, I would like to mention another important group associated to topology in three dimensions, the knot group. Consider two sets of points $\{A_1, A_2, ..., A_N\}$ and $\{B_1, B_2, ..., B_N\}$ in three dimensions, and connect them pairwisely by N ropes. We allow homotopy transformations of configurations of the ropes. Configurations of ropes natually make a group by multiplication, the knot group. There is the identity connecting A_1 to B_1, etc in the straight fashion. σ_i connects A_i to B_{i+1} and A_{i+1} to B_i. σ_i^{-1} also does so but with opposite orientation. The knot group is generated $\{\sigma_i\}_{i=1,...,N-1}$ with the two set of relations,

$$\sigma_i \sigma_k = \sigma_k \sigma_i \ (i + 1 < k), \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

Higher Homotopy Groups

A natural generalization of a loop is the n-sphere. So, we try to classify continuous maps from S^n to M. A continuous map α from the n-cube $I_n = [0,1]\times[0,1]\times\cdots\times[0,1]$ to M such that $\alpha : \partial I_n \to x_0$ is called an n-loop with base x_0. We say that two n-loops, α and β, are homotopic if there is a continuous family of n-loops $H(s)$ such that $H(s=0) = \alpha$ and $H(s=1) = \beta$. The set of homotopically equivalent classes $[\alpha]$ of n-loops defines the n-th homotopy group $\pi_n(M, x_0)$. Again, if M is archwise connected, the homotopy group can be defined without referring to the base point x_0. Higher homotopy groups are all abelian.

Homotopy Type

Homotopy of maps

A pair of continuous maps, $f, g : X \to Y$, are homotopic if there is a continuous map $H : [0,1] \times X \to Y$ such that $H(s=0) = f$ and $H(s=1) = g$.

Homotopy of spaces

Two topological spaces are of the same homotopy type iff there are continuous maps, $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is *homotopic* to the identity map on Y and $g \circ f$ is *homotopic* to the identity on X. For example, $\mathbf{R}$ is homotopic to a set consisting of one point $p \in \mathbf{R}$.

A subspace R of M is called a deformation retract of M if there is a continuous map $H : [0,1] \times M \to M$ such that $H(s=0)$ is the identity map on M, and $H(s=1)$ maps M to R and acts as the identity on R. In this case, M and R are of the same homotopy type. (R is called a strong deformation retract if $H(s)$ leaves points in R fixed for all $s \in [0,1]$.) In particular, if $p \in M$ is a deformation retract of M, then M is called contractible.

If two topological spaces are of the same homotopy type, they have the same homotopy groups.

Question 4: Show that the fundamental group of a contractible space is trivial.

Homotopy Groups of Spheres

When $0 < i < n$, any continuous map from S^i to S^n is homotopic to a constant map, and $\pi_i(S^n)$ is trivial. As for $\pi_n(S^n)$, there is the identity map $f : S^n \to S^n$, and we can then consider $[f] * [f] * \cdots * [f]$, which wrap S^n multiple times. In fact, $\pi_n(S^n) = \mathbf{Z}$, parametrized by the winding number.

More interesting is $\pi_i(S^n)$ with $i > n$. The first non-trivial example is $\pi_3(S^2)$, namely classification of maps from $S^3 \to S^2$. We have encountered this earlier in this course — the Hopf fibration. There, we regarded S^3 as the total space of fiber bundle over S^2. For a fiber bundle, there must be a projection map from the total space to the base. The Hopf fibration is characterized by an integer n, which we identified as the monopole number. In fact, $\pi_3(S^2) = \mathbf{Z}$.

Homology groups $H_i(S^n)$ for $i > n$ are all trivial. In contrast, $\pi_i(S^n)$ for $i > n$ are complex and difficult to compute.

Stable region

$$
\begin{aligned}
\pi_4(S^3) = \pi_5(S^4) = \pi_6(S^5) = \cdots = \mathbf{Z}_2, \\
\pi_6(S^4) = \pi_7(S^5) = \pi_8(S^6) = \cdots = \mathbf{Z}_2, \\
\pi_8(S^5) = \pi_9(S^6) = \pi_{10}(S^7) = \cdots = \mathbf{Z}_{24}, \\
\pi_{10}(S^6) = \pi_{11}(S^7) = \pi_{12}(S^8) = \cdots = 0, \\
\cdots .
\end{aligned} \tag{2}
$$

In general, $\pi_{2k+n}(S^{k+1+n})$ $(k = 2, 3, 4, ...; n = 0, 1, 2, 3...)$ are called stable homotopy groups of spheres, and they depend only on k.

Homotopy Groups of Groups

We will discuss topology of Lie groups in the next week. Here we will summarize some basic facts about their homotopy groups.

Suppose M is a topological space with $\pi_0(M) = \pi_1(M) = 0$. Such a space is called simply connected. Suppose that a group G acts on M, *i.e.* each $g \in G$ gives a homeomorphism on M, and that the quotient M/G is smooth. The fundamental group of the quotient is given by,

$$
\pi_1(M/G) = \pi_0(G).
$$

In particular, if G is a discrete group, $\pi_0(G) = G$ and thus $\pi_1(M/G) = G$. For example, for $M = \mathbf{R}$ and $G = \mathbf{Z}$, we have $\pi_1(S^1) = \pi_1(\mathbf{R}/\mathbf{Z}) = \mathbf{Z}$.

Bott periodicity

For the unitary group,

$$
\pi_k(U(n)) = \pi_k(SU(N)) = 0 \ (k : \text{even}), \quad \mathbf{Z} \ (k : \text{odd}),
$$

when $k > 1$ and $n \geq (n+1)/2$. This proporty was discovered by Raoul Bott and is called the Bott periodicity. It is closely related to the behavior of the homotopy groups of spheres in the stable region. There are similar periodicities for the orthogonal and symplectic groups.

It is not possible to end a lecture on homotopy without referring to the Poincare conjecture in three dimensions. Poincare initially conjectured that any three-dimensional manifold with the same betti number as those of the 3-sphere must be homeomorphic to the 3-sphere. However, he later found a counter-example, the first example of what are now called homology spheres. He then revised his conjecture as: Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

The generalized Poincare conjecture has been proven in higher dimensions. In 1961, Stephen Smale proved the conjecture for dimensions greater than four. The four-dimensional conjecture was proven by Michael Freedman.

It should be noted that the conjecture is about homeomorphism and not about diffeomorphism. For example, Milnor's exotic spheres show that the smooth Poincare' conjecture is false in dimension seven.

Finally, in 2002-2003, Grigori Perelman proved the original three-dimensional conjecture, by refinding the technique of the Ricci flow conceived by Richard Hamilton.