§7. homology 完全列

今日やること
§7 homology 完全列
§8 Mayer-Vietoris 完全列

連結準同型

$$0 \to C'_* \xrightarrow{i_*} C_* \xrightarrow{p_*} C''_* \to 0 \quad (\text{exact})$$

chain 複体の短完全列

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C'_{n+1} & \to & C_{n+1} & \to & C''_{n+1} & \to & 0 \quad (\text{exact}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C'_n & \to & C_n & \to & C''_n & \to & 0 \quad (\text{exact}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C'_{n-1} & \to & C_{n-1} & \to & C''_{n-1} & \to & 0 \quad (\text{exact}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C'_{n-2} & \to & C_{n-2} & \to & C''_{n-2} & \to & 0 \quad (\text{exact}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$\Rightarrow$

$$\partial_* : H_n(C''_*) \to H_{n-1}(C'_*) \quad \text{連結準同型}$$

$$H_n(C''_*) = \frac{\operatorname{Ker}\partial''_n}{\operatorname{Im}\partial''_{n+1}}, \qquad H_{n-1}(C'_*) = \frac{\operatorname{Ker}\partial'_{n-1}}{\operatorname{Im}\partial'_n}$$

構成法

(I) $u'' \in \operatorname{Ker}\partial''_n$ に $u' \in \operatorname{Ker}\partial'_{n-1}$ を対応させる.

(II) $u'' \in \operatorname{Ker}\partial''_n \longmapsto [u'] \in H_{n-1}(C'_*)$ が well-defined な写像である.

(III) が準同型であること.

(IV) $\partial_* : [u''] \in H_n(C''_*) \mapsto \partial_*[u''] := [u'] \in H_{n-1}(C'_*)$ が定義される.

(I)

$$\begin{tikzcd}[column sep=small]
 & \exists u \arrow[r, mapsto] \arrow[d, mapsto] & \forall u'' \arrow[d, mapsto] \\
\exists! u' \arrow[r, mapsto, "inj"] \arrow[d, mapsto] & \partial_n u \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
\partial'_{n-1} u' \arrow[r, mapsto, "inj"] & 0 & \\
0 \arrow[u, equal] & &
\end{tikzcd}$$

とり方に ambiguity あり

(II)

$$\begin{tikzcd}[column sep=small]
\exists v_1' \arrow[r, mapsto] \arrow[d, mapsto] & (\text{別の}\, u_1) - u \arrow[r, mapsto] \arrow[d, mapsto] & u'' - u'' = 0 \\
\partial_n' v_1' \arrow[r, mapsto] & \partial_n (u_1 - u) &
\end{tikzcd}$$

別の u_1 を使っても $[u' + \partial_n' v_1'] = [u'] \in H_{n-1}(C'_*)$ がでてくる。

(III) (II) により u のとり方によらないのだから

$$\begin{tikzcd}[column sep=small]
 & u + v \arrow[r, mapsto] \arrow[d, mapsto] & u'' + v'' \\
u' + v' \arrow[r, mapsto] & \partial_n u + \partial_n v &
\end{tikzcd}$$

(IV)

$$\begin{tikzcd}[column sep=small]
 & w \arrow[r, mapsto] \arrow[d, mapsto] & w'' \arrow[d, mapsto] \\
 & \partial_{n+1} w \arrow[r, mapsto] \arrow[d, mapsto] & \boxed{\partial'_{n+1} w''} \\
0 \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

//

こうしてえられた準同型

$$\partial_* : H_n(C''_*) \to H_{n-1}(C'_*)$$

は次をみたす。

<u>homology 完全列</u>

$$\cdots \xrightarrow{\partial_*} H_n(C'_*) \xrightarrow{i_*} H_n(C_*) \xrightarrow{p_*} H_n(C''_*) \xrightarrow{\partial_*} H_{n-1}(C'_*) \xrightarrow{i_*} H_{n-1}(C_*) \xrightarrow{p_*}$$

$$\cdots \longrightarrow H_0(C_*) \xrightarrow{p_*} H_0(C''_*) \longrightarrow 0 \quad (\text{exact})$$

(1) $H_n(C_*)$ における完全性

$$\mathrm{Im}\, i_* \subset \mathrm{Ker}\, p_* \Leftrightarrow p_* i_* = 0 \Leftarrow p_n i_n = 0$$

$$\mathrm{Im}\, i_* \supset \mathrm{Ker}\, p_* \ni [u]$$

$$\begin{array}{ccccc} & & v & \mapsto & v'' \\ & & \downarrow & & \downarrow \\ & & \partial v & \mapsto & 0 \\ \exists\, u' & \mapsto & u - \partial v & & \\ & & u & \mapsto & pu \\ \downarrow & & \downarrow & & \\ \partial u' & \xrightarrow{inj} & 0 & & \\ \| & & & & \\ 0 & & & & \end{array}$$

(2) $H_n(C''_*)$ における完全性

$$\mathrm{Im}\, p_* \subset \mathrm{Ker}\, \partial_*$$

$$\begin{array}{ccc} & u & \mapsto pu \\ & \downarrow & \\ 0 \mapsto & 0 & \end{array}$$

$$\mathrm{Im}\, p_* \supset \mathrm{Ker}\, \partial_*$$

$$\begin{array}{ccccc} v' & \mapsto & u - iv' & & \\ & & u & \mapsto & u'' \\ \downarrow & & \downarrow & & \downarrow \\ u' & \mapsto & \partial u & & 0 \end{array}$$

(3) $H_{n-1}(C'_*)$ における完全性

$$\mathrm{Im}\, \partial_* \subset \mathrm{Ker}\, i_*$$

$$\begin{array}{ccc} & u & \mapsto u'' \\ & \downarrow & \\ u' \mapsto & \partial u & \end{array}$$

$$i_*[u'] = [\partial u] = 0 \in H_{n-1}(C_*)$$

$$\mathrm{Im}\, \partial_* \supset \mathrm{Ker}\, i_* \ni [u']$$

$$\begin{array}{ccc} & u & \mapsto pu \\ & \downarrow & \\ u' \mapsto & iu' & \end{array}$$

(4) $H_0(C''_*)$ における完全性

$n<0$ について $C_n = C'_n = C''_n = 0$ とみれば

$$H_0(C_*) \to H_0(C''_*) \to \underset{\substack{\| \\ 0}}{H_{-1}(C'_*)} \quad (\text{exact})$$

連結準同型の自然性

何に対して自然か？

$\Downarrow$

chain 複体の短完全列の準同型

$$
\begin{tikzcd}
0 \arrow[r] & C'_* \arrow[r, "p_*"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & C_* \arrow[r, "i_*"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & C''_* \arrow[r] \arrow[d, "f_*"] & 0 \\
0 \arrow[r] & D'_* \arrow[r, "\varpi_*"'] & D_* \arrow[r, "\iota_*"'] & D''_* \arrow[r] & 0
\end{tikzcd}
$$

各 n で可換ということ

$\Downarrow$

$$
\begin{tikzcd}
H_n(C''_*) \arrow[r, "\partial_*"] \arrow[d, "f''_*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(C'_*) \arrow[d, "f'_*"] \\
H_n(D''_*) \arrow[r, "\partial_*"'] & H_{n-1}(D'_*)
\end{tikzcd}
$$

(証)

$$
\begin{array}{ccc}
 & \exists u & \longmapsto \forall u'' \\
 & \downarrow & \\
u' & \longmapsto \partial u &
\end{array}
\quad \xrightarrow{f_*} \quad
\begin{array}{ccc}
 & f_* u & \longmapsto f''_* u'' \\
 & \downarrow & \\
\partial' f'_* u' & \longmapsto \partial f_* u &
\end{array}
\quad //
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & H_n(C_*) \arrow[r, "p_*"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n(C''_*) \arrow[r, "\partial_*"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(C'_*) \arrow[r, "i_*"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(C_*) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\arrow[r] & H_n(D_*) \arrow[r, "\varpi_*"'] & H_n(D''_*) \arrow[r, "\partial_*"'] & H_{n-1}(D'_*) \arrow[r, "\iota_*"'] & H_{n-1}(D_*) \arrow[r] & \cdots
\end{tikzcd}
$$

「長完全列の準同型」がでてくる

(つまり homology 完全列の自然性)

まとめ：定理 7.1

応用例

命題7.2.

6つとも複体で
うち5つがexact
⇒ のこり一つも
exactになる.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_2 & \to & B_2 & \to & C_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_1 & \to & B_1 & \to & C_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_0 & \to & B_0 & \to & C_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(証) 縦のどれかがexactかどうかわからないとしてよい

⇒ homology 完全列

$$0 \to H_*(\text{exactかどうかわからない}) \to 0 \quad (\text{exact})$$

これも $= 0$ //

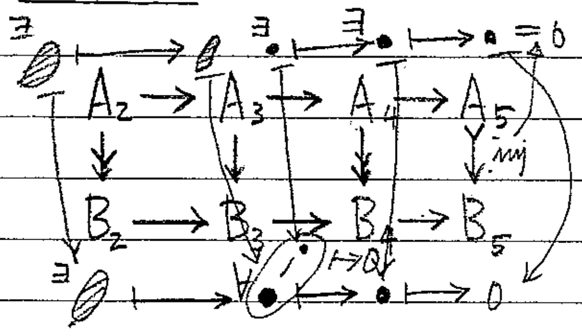
別のdiagram chase

補題7.3. (5-lemma)

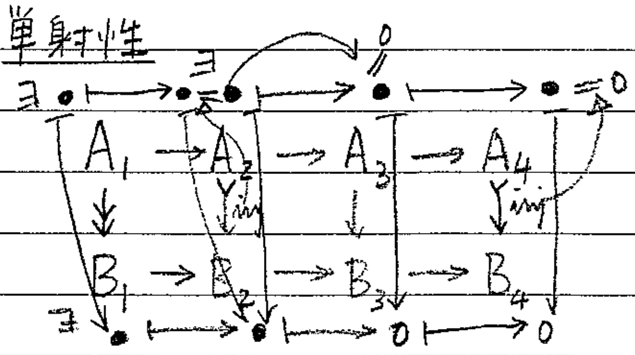
$$\begin{array}{ccccccccccc}
A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 & (\text{exact}) & \\
\downarrow\cong & \circlearrowleft & \downarrow\cong & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow\cong & \circlearrowleft & \downarrow\cong & & \\
B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 & (\text{exact}) &
\end{array}
\quad \Rightarrow \quad
\begin{array}{c} A_3 \\ \downarrow\cong \\ B_3 \end{array}$$

(証)

全射性



単射性



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No. Date . .

§8 Mayer-Vietoris 完全列

<u>定理 7.1. (Mayer-Vietoris 完全列)</u>

X：位相空間　$U, V \subset X$ 部分空間

$\mathring{U} \cup \mathring{V} = X$　(ただし、$\mathring{U}, \mathring{V}$ は X における U, V の内部)

$\Longrightarrow$　$\partial_* : H_n(X) \to H_{n-1}(U \cap V)$ 連結準同型

$$\xrightarrow{i} H_n(U) \oplus H_n(V) \xrightarrow{j} H_n(X) \xrightarrow{\partial_*} H_{n-1}(U \cap V) \xrightarrow{i} H_{n-1}(U) \oplus H_{n-1}(V) \xrightarrow{j}$$

$$(u, v) \longmapsto j_{U*}u + j_{V*}v \qquad w \longmapsto (i_{U*}w, -i_{V*}w)$$

$j_U : U \hookrightarrow X$, $j_V : V \hookrightarrow X$ inclusion　　$i_U : U \cap V \hookrightarrow V$, $i_V : U \cap V \hookrightarrow U$ inclusion

$$\cdots \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0 \quad (\text{exact})$$

<u>自然性</u>　Y：位相空間，$U', V' \subset Y$，$\mathring{U}' \cup \mathring{V}' = Y$

$f : X \to Y$ 連続写像

$f(U) \subset U'$，$f(V) \subset V'$

$\Longrightarrow$

$$\begin{tikzcd} H_n(X) \arrow[r, "\partial_*"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & H_{n-1}(U \cap V) \arrow[d, "f_*"] \\ H_n(Y) \arrow[r, "\partial_*"] & H_{n-1}(U' \cap V') \end{tikzcd}$$

<u>定理の証明のあらすじ</u>

$$\begin{tikzcd} 0 \arrow[r] & S_*(U \cap V) \arrow[r, "i"] & S_*(U) \oplus S_*(V) \arrow[r, "j"] & S_*(U) + S_*(V) \arrow[r] \arrow[d, "\text{IS}"] & 0 \quad (\text{exact}) \\ & & & S_*(X) & \end{tikzcd}$$

ここが肝要

$\Longrightarrow$

$$\begin{tikzcd} \cdots \arrow[r] & H_n(U \cap V) \arrow[r, "i"] & H_n(U) \oplus H_n(V) \arrow[r, "j"] & H_n(S_*(U) + S_*(V)) \arrow[r, "\partial_*"] \arrow[d, "\cong"] & \cdots \\ & & & H_n(X) & \end{tikzcd}$$

$\Longrightarrow$ 自然性は明らか

$n \geq 0$.

$$S_n(X) = \mathbb{Z}(X^{\Delta^n})$$

$$S_n(U),\ S_n(V),\ S_n(U\cap V) \hookrightarrow S_n(X)$$

$$U^{\Delta^n} \cap V^{\Delta^n} = \{\sigma : \Delta^n \to X \;;\; \sigma(\Delta^n) \subset U \text{ かつ } \sigma(\Delta^n) \subset V\}$$
$$= \{\sigma : \Delta^n \to X \;;\; \sigma(\Delta^n) \subset U\cap V\}$$
$$= (U\cap V)^{\Delta^n}$$

$$S_n(U) \cap S_n(V) = \mathbb{Z}U^{\Delta^n} \cap \mathbb{Z}V^{\Delta^n} \underset{\text{Lem 2.3(1)}}{=} \mathbb{Z}(U\cap V)^{\Delta^n} = S_n(U\cap V)$$

かくして Lem 2.7 より

$$0 \to S_*(U\cap V) \to S_*(U)\oplus S_*(V) \to S_*(U)+S_*(V) \to 0 \text{ (exa}$$

$$S_*(U)+S_*(V) \underset{?}{\overset{\cong}{\hookrightarrow}} S_*(X)$$

定理8.2. $\mathring{U} \cup \mathring{V} = X$

$$\Rightarrow \forall n \geq 0 \quad H_n(S_*(U)+S_*(V)) \xrightarrow{\cong} H_n(X)$$

証明のあらすじ

全射 $\forall u \in Z_n(S_*(X)) \subset S_n(X)$

$$u = \sum_{l=1}^{L} a_l\sigma_l,\quad a_l \in \mathbb{Z},\ \sigma_l : \Delta^n \to X \text{ conti. map}$$

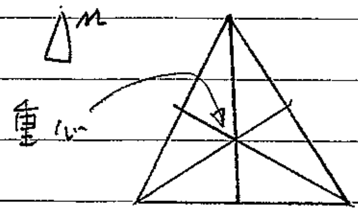
$\{\sigma_l^{-1}(\mathring{U}),\ \sigma_l^{-1}(\mathring{V})\}$: Δ^n (compact metric space) の開被覆

Lebesgue 数 $\exists \rho_l > 0$

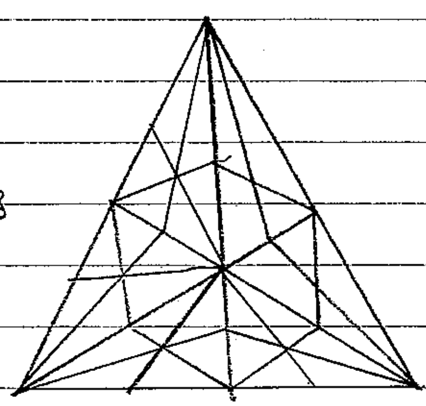
$$A \subset \Delta^n,\ \delta(A) \leq \rho_l \Rightarrow \sigma_l(A) \subset U \text{ または } \sigma_l(A) \subset V$$

重心細分 (pp16-)

Δ^n



重心細分すると
直径が $\leq \frac{n}{n+1}$ 倍になる
(Lem 8.11)



くりかえし細分すると 直径 $\leqq \rho_\ell$ とできる

$\Downarrow$

$v_\ell = ($ σ_ℓ の細分でえられた pieces を適当な符号をつけて足し上げたもの $)$

$\in S_n(U) + S_n(V)$

$[u] = [\Sigma a_\ell \sigma_\ell] = [\Sigma a_\ell v_\ell] \in \mathrm{Im}(H_n(S_*(U)+S_*(V)) \to H_n(X))$

$\Longrightarrow$ 全射性

<u>単射性</u> $H_n(S_*(U)+S_*(V)) \to H_n(X)$

$[v] \longmapsto 0$ とする

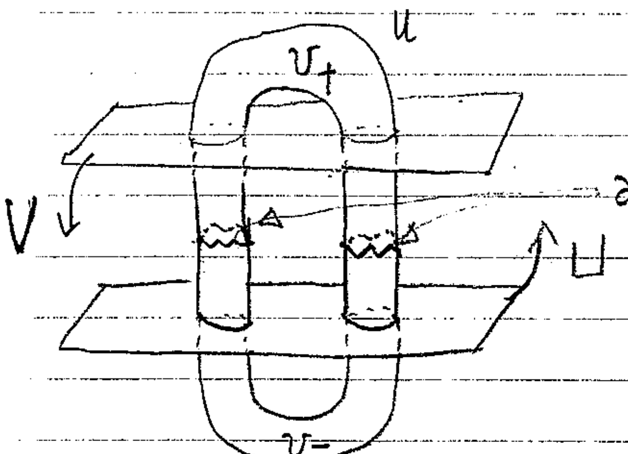
$\exists w \in S_{n+1}(X),\ v = \partial w$

wを充分細かく砕くと $\exists w' \in S_{n+1}(U) + S_{n+1}(V)$

$\partial w' = \partial w = v$ ゆえに $[v] = 0 \in H_n(S_*(U)+S_*(V))$

$\Longrightarrow$ 単射性 //

<u>連結準同型の幾何学的意味</u>



$\partial v_+ = -\partial v_-$

$[u] = [v_+ + v_-] \in H_n(X)$

$0 = \partial(v_+ + v_-) = \partial v_+ + \partial v_-$

$\partial v_- = -\partial v_+$

$$\begin{array}{ccc} & & S_n(U)\oplus S_n(V) \to S_n(U)+S_n(V) \\ & & (v_+, v_-) \longmapsto v_+ + v_- \\ & & \downarrow \\ S_{n-1}(U\cap V) & \to & S_{n-1}(U)\oplus S_{n-1}(V) \\ \partial v_+ & \longmapsto & (\partial v_+, \partial v_-) \\ & & \| \\ & & (\partial v_+, -\partial v_+) \end{array}$$

$\partial_*[u] = [\partial v_+] \in H_{n-1}(U\cap V)$

$\partial_* = ($ UとVを分ける「切り口」を見る写像 $)$

$\rightsquigarrow$ 完全性の証明をなぞってみる.

被約 homology 群

X：位相空間 $\neq \phi$

添加複体

$$\tilde{S}_*(X) : \cdots \to \overset{n}{S_n(X)} \to \overset{n-1}{S_{n-1}(X)} \to \cdots$$

$$\cdots \to \overset{1}{S_1(X)} \to \overset{0}{S_0(X)} \xrightarrow{\varepsilon} \overset{-1}{\mathbb{Z}} \to 0$$

$$\sum a_j \sigma_j \longmapsto \sum a_j$$

ε：全射 $\Leftarrow X \neq \phi$

ゆえに $\tilde{H}_{-1}(X) = 0$

$$\tilde{H}_*(X) \overset{def}{=} H_*(\tilde{S}_*(X))$$

$$\underset{Lem 7.3}{=} \begin{cases} H_n(X) & \text{if } n \geq 1 \\ \mathrm{Ker}(\varepsilon : \mathbb{Z}\pi_0(X) \to \mathbb{Z}) & \text{if } n = 0 \\ 0 & \text{if } n \leq -1 \end{cases}$$

とくに X : path-conn $\Leftrightarrow$ $\tilde{H}_0(X) = 0$

定理 8.4. X, U, V 上述の通り，さらに $U \cap V \neq \phi$ とする

$\Rightarrow$ $\cdots \to H_1(X) \to \tilde{H}_0(U \cap V) \to \tilde{H}_0(U) \oplus \tilde{H}_0(V) \to \tilde{H}_0(X) \to 0$ (exact)

(pf) $0 \to \tilde{S}_*(U \cap V) \to \tilde{S}_*(U) \oplus \tilde{S}_*(V) \to \tilde{S}_*(U) + \tilde{S}_*(V) \to 0$ (exact)

$$\tilde{S}_*(U) + \tilde{S}_*(V) \underset{\simeq}{\hookrightarrow} \tilde{S}_*(X)$$

//