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§5 特異 homology 群の定義

商群の普遍性（「商群がわかれば数学初段」（岩堀長慶））

M：可換群（加法的 $+$）　$\mathbb{Z}$ 加群

$N < M$：部分(加)群

$M/N := \{x+N \,;\, x \in M\} \subset$（$M$ の巾集合）

$(x+N)+(y+N) := (x+y)+N$　well-defined

$\Longrightarrow$ 商群

$p : M \to M/N,\ x \mapsto x+N$,　natural canonical projection

補題 5.1（商群の普遍性）

$\forall L$：可換群

$\forall f : M \to L$ homom

$f|_N = 0$

$$\begin{tikzcd} & M \arrow[dr, "p"] \arrow[d, "\forall f"] & \\ N \arrow[ur] \arrow[r, "0"'] & L & M/N \arrow[l, dashed, "\exists ! \bar{f}"] \end{tikzcd}$$

$\Longrightarrow \exists ! \bar{f} : M/N \to L$ homom. s.t. $f = \bar{f} \circ p : M \to L$

いいかえ

$$0 \to \mathrm{Hom}(M/N, L) \xrightarrow{\circ p} \mathrm{Hom}(M, L) \xrightarrow{|_N} \mathrm{Hom}(N, L)$$

(exact)

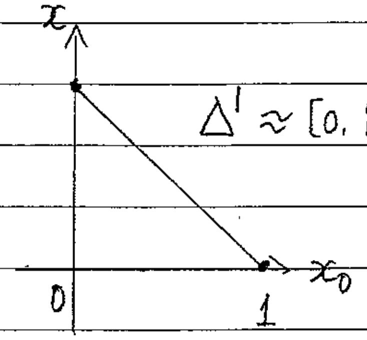
標準 n 単体（standard n-simplex）Δ^n

$n \geqq 0$

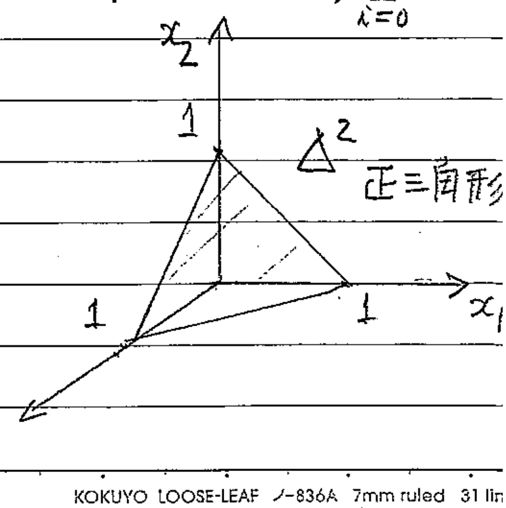
$$\Delta^n := \{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \,;\, x_i \geqq 0\ (0 \leqq \forall i \leqq n),\ \sum_{i=0}^{n} x_i = 1\}$$

凸集合

$\Delta^0 = \{1\} \subset \mathbb{R}$



$\Delta^1 \approx [0, 1]$



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面写像 (face map)

$d_i = d_i^{n-1} : \Delta^{n-1} \to \Delta^n$, $0 \leq i \leq n$, face map

$d_i(x_0, x_1, \cdots, x_{n-1}) := (x_0, \cdots, x_{i-1}, 0, x_i, \cdots, x_{n-1})$

$d_i(\Delta^{n-1}) \subset \Delta^n$ ($d_i : \mathbb{R}^n \to \mathbb{R}^{n+1}$ linear map)

補題5.2 $d_i d_j = d_j d_{i-1} : \Delta^{n-1} \to \Delta^{n+1}$ if $i > j$

($e_i^{n-1} := (\overset{0}{0}, \cdots, \overset{i-1}{0}, \overset{i}{1}, \overset{i+1}{0}, \cdots, \overset{n-1}{0}) \in \Delta^{n-1} \subset \mathbb{R}^n$
の上で両辺の値が一致することを直接計算する (プリント pp 2-3))

特異chain複体

X, Z : 位相空間

$X^Z := \{\sigma : Z \to X$: 連続写像$\}$

$n \geq 0$

$S_n(X) := \mathbb{Z}X^{\Delta^n}$. $X^{\Delta^n} = \{\sigma : \Delta^n \to X$ conti map$\}$ の生成する自由加群

$\partial_n : S_n(X) \to S_{n-1}(X)$

$\partial_n \sigma := \sum_{i=0}^{n} (-1)^i \underbrace{(\sigma \circ d_i^{n-1})}_{\in X^{\Delta^{n-1}}} \in S_{n-1}(X)$ $(\sigma \in X^{\Delta^n})$

$S_*(X) := \{S_n(X), \partial_n\}_{n \geq 0}$: 位相空間Xの特異chain複体

補題5.3 $\forall n \geq 1$ $\partial_n \partial_{n+1} = 0 : S_{n+1}(X) \to S_{n-1}(X)$

(証) $\forall \sigma \in X^{\Delta^{n+1}}$ $\partial_n \partial_{n+1} \sigma = 0$ を示せばよい

$\partial_n \partial_{n+1} \sigma = \partial_n \left(\sum_{i=0}^{n+1} (-1)^i \sigma d_i \right) = \sum_{i=0}^{n+1} (-1)^i \partial_n (\sigma d_i)$

$= \sum_{i=0}^{n+1} \sum_{j=0}^{n} (-1)^{i+j} \sigma d_i d_j = \sum_{j<i} (-1)^{i+j} \sigma d_i d_j + \sum_{i \leq j} (-1)^{i+j} \sigma d_i d_j$

$$\underset{\text{Lem 5.2}}{=} \sum_{j<i\leqq n+1} (-1)^{i+j} \sigma d_j d_{i-1} + \sum_{i\leqq j\leqq n} (-1)^{i+j} \sigma d_i d_j$$

($k=j,\ l=i-1$ とおく)

$$= \sum_{k\leqq l\leqq n} (-1)^{k+l+1} \sigma d_k d_l + \sum_{i\leqq j\leqq n} (-1)^{i+j} \sigma d_i d_j$$

$= 0$ //

Chain 複体

定義 $C_* = \{C_q, \partial_q : C_q \to C_{q-1}\}_{q\geq 0}$: chain complex

$\overset{\text{def}}{\Longleftrightarrow}$ 0) C_q : $\mathbb{Z}$-module ($q \geq 0$)

$\partial_q : C_q \to C_{q-1}$ homom (boundary map とよぶ)

(ただし $C_{-1} = 0$, $\partial_0 = 0$ と理解する)

1) $\partial_q \partial_{q+1} = 0 : C_{q+1} \to C_{q-1}$ ($\forall q \geq 1$)

記号 $Z_q(C_*) := \mathrm{Ker}(\partial_q : C_q \to C_{q-1}) \subset C_q$

$\quad \ni$ q-cycle

$B_q(C_*) := \mathrm{Im}(\partial_{q+1} : C_{q+1} \to C_q) \subset C_q$

$\quad \ni$ q-boundary.

$\partial_q \partial_{q+1} = 0$

$\Longleftrightarrow 0 = \partial_q \partial_{q+1}(C_q) = \partial_q B_q(C_*)$

$\Longleftrightarrow B_q(C_*) \subset Z_q(C_*)$

$$H_q(C_*) \overset{\text{def}}{=} Z_q(C_*) / B_q(C_*) = \mathrm{Ker}\,\partial_q / \mathrm{Im}\,\partial_{q+1}$$

the q^{th} homology group of C_*

No. Date . .

$u \in Z_q(C_*) = \mathrm{Ker}\,\partial_q$

$[u] := u + B_q(C_*) \in H_q(C_*)$ homology class of u

$H_*(C_*) := \{H_q(C_*)\}_{q \geq 0}$ 次数加群

$q=0$ のとき

$Z_0(C_*) = C_0$

$H_0(C_*) = C_0 / B_0(C_*) = C_0/\partial_1 C_1 = \mathrm{Coker}\,\partial_1$

補題 5.4 $C_* : \to C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \to \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ 列

(a) C_* : chain cpx かつ $H_q(C_*) = 0$ $(\forall q \geq 1)$

$\Updownarrow$

(b) C_* : exact

(証) 各 $q \geq 1$ について

C_q において exact $\Leftrightarrow$ $\mathrm{Ker}\,\partial_q = \mathrm{Im}\,\partial_{q+1}$ $\Leftrightarrow$ $H_q(C_*) = 0$ //

X : 位相空間

$S_*(X)$: chain complex ($\Leftarrow$ Lem 5.3)

$H_q(X) = H_q(X;\mathbb{Z}) \overset{\text{def}}{=} H_q(S_*(X))$ $(q \geq 0)$

$= \mathrm{Ker}(\partial_q : S_q(X) \to S_{q-1}(X)) / \partial_{q+1}(S_{q+1}(X))$

X の q 次元整係数特異 homology 群

$H_*(X) := H_*(S_*(X))$ X の特異 homology 群

背景にあるもの

X : C^∞ mfd

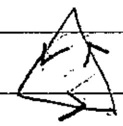
ω : n-form on X

$\sigma : \Delta^n \to X$ C^∞ map

$\int_\sigma \omega \left(= \int_{\Delta^n} \sigma^* \omega\right) \in \mathbb{R}$ が考えられる

Stokes' Thm θ: $(n-1)$-form on X

$$\int_\sigma d\theta = \int_{\partial\sigma}\theta = \sum_{i=0}^{n}(-1)^i\int_{\sigma\circ d_i}\theta$$



C^∞ structure, differential forms, ··· 全部忘れて抽象化
$\partial\sigma$ だけ覚えておく $\Longrightarrow$ 特異 homology 群

補題 5.5 (基本性質 (IV))

$$H_q(*) = \begin{cases} \mathbb{Z} & \text{if } q=0 \\ 0 & \text{if } q\neq 0 \end{cases}$$

(証) 連続写像 $\Delta^q \to *$ は一つしかない
$f_q : \Delta^q \to *$, $x \mapsto *$, と表す

$$S_q(*) = \mathbb{Z}\,*^{\Delta^q} = \mathbb{Z}\{f_q\} \cong \mathbb{Z}$$

$$\partial_q f_q = \sum_{i=0}^{q}(-1)^i f_q\circ d_i = \sum_{i=0}^{q}(-1)^i f_{q-1}$$

$$= \begin{cases} 0 & \text{if } q \text{ is odd} \\ f_{q-1} & \text{if } q \text{ is even} \end{cases}$$

$$S_*(*) : \xrightarrow{1} \overset{2n-1}{\mathbb{Z}} \xrightarrow{0} \overset{2n}{\mathbb{Z}} \xrightarrow{1} \mathbb{Z} \longrightarrow \cdots \xrightarrow{1} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}}$$

これは exact

$$H_q(*) = 0 \quad \text{if } q \geq 1$$
$$H_0(*) = \mathbb{Z} \quad //$$

連続写像との関係

X, Y : top. sp's $\quad f : X \to Y$ conti. map

$$f_* : S_n(X) = \mathbb{Z}X^{\Delta^n} \to S_n(Y) = \mathbb{Z}Y^{\Delta^n}$$

$$(\sigma : \Delta^n \to X) \longmapsto (f\circ\sigma : \Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y)$$

(5.3) $\partial_n \circ f_* = f_* \circ \partial_n : S_n(X) \to S_{n-1}(Y)$

(∵) $\partial_n f_*(\sigma) = \partial_n(f \circ \sigma) = \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ d_i$
$= f_*\left(\sum_{i=0}^{n} (-1)^i \sigma \circ d_i\right) = f_* \partial_n \sigma$ //

(5.3) をみたす homom の列 = chain map

Chain map

$C_* = \{C_q, \partial_q : C_q \to C_{q-1}\}_{q \geq 0}$ chain complexes

$D_* = \{D_q, \partial'_q : D_q \to D_{q-1}\}_{q \geq 0}$

定義 $f_* : C_* \to D_*$ chain map

$\overset{\text{def}}{\iff}$ 0) $f_* = \{f_* : C_q \to D_q\}_{q \geq 0}$ homom のあつまり

1)
$$\begin{tikzcd}
\cdots \arrow[r] & C_{q+1} \arrow[r, "\partial_{q+1}"] \arrow[d, "f_*"] & C_q \arrow[r, "\partial_q"] \arrow[d, "f_*"] & C_{q-1} \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & D_{q+1} \arrow[r, "\partial'_{q+1}"'] & D_q \arrow[r, "\partial'_q"'] & D_{q-1} \arrow[r] & \cdots
\end{tikzcd}$$

このとき

(5.5) $f_*(Z_q(C_*)) \subset Z_q(D_*)$

(∵) $\forall u \in Z_q(C_*)$, $\partial'_q f_*(u) = f_* \partial_q(u) = f_* 0 = 0$
$f_* u \in \operatorname{Ker} \partial'_q = Z_q(D_*)$ //

(5.6) $f_*(B_q(C_*)) \subset B_q(D_*)$

(∵) $\forall u \in B_q(C_*) = \partial_{q+1} C_{q+1}$, $\exists v \in C_{q+1}$, $u = \partial_{q+1}(v)$
$f_* u = f_* \partial_{q+1} v = \partial'_{q+1} f_* v \in B_q(D_*)$ //

$$\begin{array}{ccccc}
u & \longmapsto & & & f_*(u) \\
Z_q(C_*) & \xrightarrow{\quad f_* \quad} & Z_q(D_*) & & \\
\downarrow & \circlearrowleft & \downarrow & & \\
[u] \in Z_q(C_*)/B_q(C_*) & \overset{\exists!\, f_*}{\dashrightarrow} & Z_q(D_*)/B_q(D_*) & & [f_* u] \\
\| & \text{induced homom} & \| & & \\
H_q(C_*) & & H_q(D_*) & &
\end{array}$$

これは $B_q(C_*)$ 上 0 になる
⇑
(5.6)

つまり $f_*[u] := [f_*(u)]$ は well-defined

位相空間にもどる

X, Y : top. sp's

$f : X \to Y$ conti. map

$\Longrightarrow f_* : S_*(X) \to S_*(Y)$ chain map

$\Longrightarrow f_* : H_q(X) \to H_q(Y)$ induced homom

$[z] \longmapsto [f_* z]$

補題 5.6 (特異 homology 群の函手性)

(1) $1_{X*} = 1_{H_q(X)} : H_q(X) \to H_q(X)$

(2)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Z \end{tikzcd} \Longrightarrow \begin{tikzcd} H_q(X) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & H_q(Y) \arrow[d, "g_*"] \arrow[dl, phantom, "\circlearrowleft"] \\ & H_q(Z) \end{tikzcd}$$

(定義から明らか)

H_q : (位相空間) $\longrightarrow$ ($\mathbb{Z}$-module) 共変函手

補題 5.7 「性質(III)」

$X_\lambda \subset X$ $\lambda \in \pi_0(X)$ に対応する弧状連結成分

$\Longrightarrow \forall q \geq 0 \quad H_q(X) = \bigoplus_{\lambda \in \pi_0(X)} H_q(X_\lambda)$

(証) Δ^q : path-conn

$\forall \sigma \in X^{\Delta^n}\ \exists! \lambda \in \pi_0(X)\quad \sigma(\Delta^n) \subset X_\lambda$

$$S_*(X) = \bigoplus_{\lambda \in \pi_0(X)} S_*(X_\lambda)$$ chain cpx の直和分解

$$\partial_n(S_n(X_\lambda)) \subset S_{n-1}(X_\lambda)$$

$$H_*(X) = \bigoplus_{\lambda \in \pi_0(X)} H_*(X_\lambda).$$ //

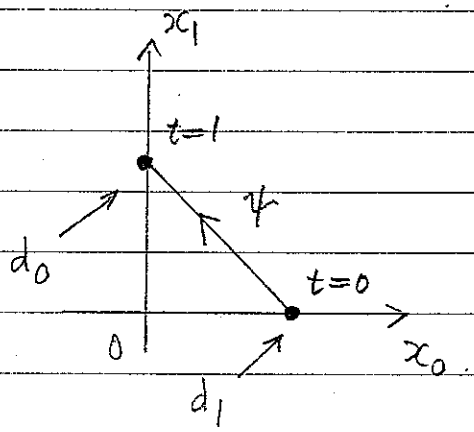
補題 5.8. 「性質(IV)」

$$H_0(X) = \mathbb{Z}\pi_0(X)$$ 自然な同型

(証) $\Delta^0 = \{1\} \subset \mathbb{R}$

$\psi : I = [0,1] \xrightarrow{\approx} \Delta^1$

$t \longmapsto (1-t, t)$



$$\begin{tikzcd}
\sigma \in \mathbb{Z}X^{\Delta^1} \arrow[r, "\partial_1"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z}X^{\Delta^0} \ni \rho \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & H_0(X) \arrow[r] \arrow[d, dashed, "\cong\ \exists!"] & 0 \quad (\text{exact}) \\
\sigma\circ\psi \in \mathbb{Z}X^{I} \arrow[r, "D_X"] & \mathbb{Z}X \ni \rho(1) \arrow[r, "\varpi_X"] & \mathbb{Z}\pi_0(X) \arrow[r] & 0 \quad (\text{exact})(\Leftarrow \text{Lem } 2.8)
\end{tikzcd}$$

$$\begin{array}{ccc}
\sigma & \longmapsto & \sigma\circ d_0 - \sigma\circ d_1 \\
\Big\downarrow & & \Big\downarrow \\
 & & \sigma(0,1) - \sigma(1,0) \\
 & & \| \\
\sigma\circ\psi & \longmapsto & \sigma\psi(1) - \sigma\psi(0)
\end{array}$$ //