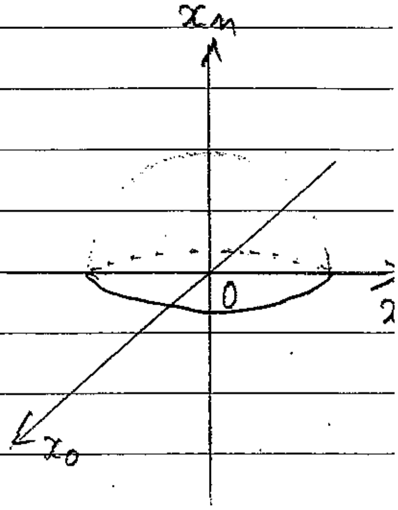
§4 球面の写像度

$n \geqq 1$

$S^n = \{(x_0, x_1, \cdots x_n) \in \mathbb{R}^{n+1} ; \sum_{i=0}^{n} x_i^2 = 1\}$



Th3.6. $H_q(S^n) = \begin{cases} \mathbb{Z} & \text{if } q = 0, n \\ 0 & \text{otherwise} \end{cases}$

応用 · Brouwer の不動点定理

· mapping degree

· $S^1 = \{z \in \mathbb{C} : |z| = 1\}$,

$a \in \mathbb{Z}$

$\varphi_a : S^1 \to S^1, \quad z \mapsto z^a$

$\varphi_1 = 1_{S^1}$

$\varphi_{a*} = ? : H_1(S^1) \to H_1(S^1)$

· $\mathbb{C}P^1 \approx S^2$ 代数学の基本定理

Brouwer の不動点定理

Retract

X : top.sp. $A \subset X$ subsp. $i : A \hookrightarrow X$ inclusion

定義 A : X の retract

$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \exists r : X \to A \text{ conti. map} \\ \text{s.t. } r \circ i = 1_A : A \to A. \end{cases}$

r : retraction

例 変位retract $\underset{\not\Leftarrow}{\Rightarrow}$ retract

· $\phi \neq X \neq *$ $x_0 \in X$

$\Rightarrow$ $\{x_0\} \subset X$ retract だが 変位retract ではない

No. Date

補題 4.1 $A \subset X$ retract, $i : A \hookrightarrow X$ inclusion
$\Longrightarrow \forall q \geq 0 \quad i_* : H_q(A) \to H_q(X)$ injective

(pf) $r : X \to A$ retraction

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, equal] & X \arrow[d, "r"] \\ & A \end{tikzcd} \Longrightarrow \begin{tikzcd} H_q(A) \arrow[r, "i_*"] \arrow[dr, equal] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & H_q(X) \arrow[d, "r_*"] \\ & H_q(A) \end{tikzcd}$$

$u \longmapsto 0$ とすると

$u = 0$ となる //

$D^n := \{ (x_1, \cdots x_n) \in \mathbb{R}^n ; \sum_{i=1}^{n} x_i^2 \leqq 1 \}$

$\partial D^n = S^{n-1} = \{ (x_1, \cdots, x_n) \in \mathbb{R}^n ; \sum_{i=1}^{n} x_i^2 = 1 \} \subset D^n$

命題 4.2 $\partial D^n \subset D^n$ retract でない

(証) $\underline{n=1}$ $\quad H_0(\partial D^1) \to H_0(D^1)$ 単射でない

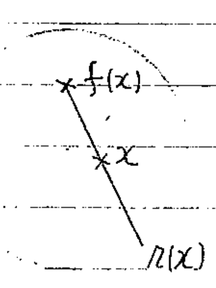
$$\begin{array}{ccc} H_0(\partial D^1) & \to & H_0(D^1) \\ \| & & \| \\ \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \end{array}$$

$\underline{n \geqq 2}$ $\quad H_{n-1}(\partial D^n) \to H_{n-1}(D^n)$ 単射でない //

$$\begin{array}{ccc} H_{n-1}(\partial D^n) & \to & H_{n-1}(D^n) \\ \| & & \| \\ \mathbb{Z} & & 0 \end{array}$$

定理 4.3 (Brouwer の不動点定理)
$\forall f : D^n \to D^n$ conti map $\exists x \in D^n \quad f(x) = x$ (不動点をもつ)

(証) 背理法 $\forall x \in D^n \quad f(x) \neq x$ と仮定する

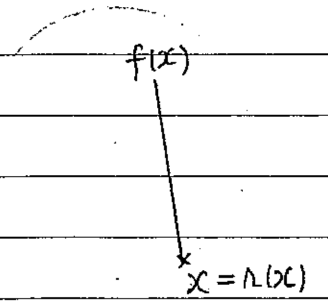


$r(x) = \begin{pmatrix} f(x) \text{ と } x \text{ をむすぶ直線と} \\ \partial D^n \text{ の交点のうち } x \text{ に近い方} \end{pmatrix} \in \partial D^n$

$r : D^n \to \partial D^n$ well defined かつ 連続

well defined $\Uparrow$ 背理法の仮定

連続 $\Uparrow$ (プリント)



$\forall x \in \partial D^n,\ r(x) = x$

つまり

$r \circ i = 1_{\partial D^n} : \partial D^n \to \partial D^n$

$r : D^n \to \partial D^n$ retraction 矛盾 //

写像度

$n \geq 1$

$H_n(S^n) = \mathbb{Z}$

$f : S^n \to S^n$ 連続写像

$$\begin{array}{ccccc} f_* : & H_n(S^n) & \to & H_n(S^n) \\ & \wr\wr & & \wr\wr \\ & \mathbb{Z} & & \mathbb{Z} \\ & \Psi & & \Psi \\ & 1 & \longmapsto & ? \end{array}$$

$\exists ! d \in \mathbb{Z} \quad \forall u \in H_n(S^n) \quad f_*(u) = du$

$\deg(f) := d \in \mathbb{Z}$ mapping degree of f

補題4.4 $f, g : S^n \to S^n$ conti. maps

(1) $\deg(1_{S^n}) = 1$

(2) $\deg(g \circ f) = (\deg g)(\deg f)$

(3) $f : S^n \to S^n$: homeo

$\Rightarrow \deg(f) = \pm 1$

(4) $\deg(f) \neq 0$

$\Rightarrow f : S^n \to S^n$ 全射

(5) $f \simeq g : S^n \to S^n$

$\Rightarrow \deg(f) = \deg(g)$

(証) (1)(2) homology群の函手性

(3) $1 = \deg(1_{S^n}) = \deg(f^{-1} \circ f) = \deg(f^{-1})\deg(f)$

$\deg f,\ \deg(f^{-1}) \in \mathbb{Z}$ より

$\deg f = \deg(f^{-1}) = \pm 1$

(4) 対偶 f：全射でない $\Rightarrow \deg(f)=0$
を示す　$\exists x \in S^n \setminus f(S^n)$

$$\begin{array}{ccccc} S^n & \xrightarrow{f} & S^n\setminus\{x\} & \hookrightarrow & S^n \\ & & \cong & & \\ & & \mathbb{R}^n \simeq * & & \end{array}$$

$H_n \Downarrow$

$$H_n(S^n) \to H_n(*) \to H_n(S^n)$$
$$H_n(*) = 0$$

ゆえに $\deg f = 0$ //

S^1 についての計算

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$

$a \in \mathbb{Z}$

$\varphi_a : S^1 \to S^1,\ z \mapsto z^a$

$\varphi_1 = 1_{S^n}$, $\varphi_0 = \mathrm{const}$

目標　<u>定理4.5.</u>　$\deg(\varphi_a) = a$

co H-space

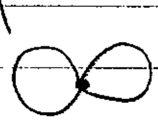
<u>定義</u> (X, x_0) : pointed space

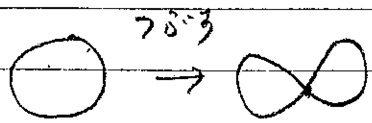
$\overset{\text{def}}{\Longleftrightarrow}$
1) X : top. sp
2) $x_0 \in X$ (basepoint という)

以下 $1 \in S^1$ を base point とする

$S^1 \vee S^1 := \{(z,w) \in S^1 \times S^1 : z=1$ または $w=1\}$

$\|$





$\mu : S^1 \to S^1 \vee S^1$

$$\mu(e^{2\pi\sqrt{-1}t}) := \begin{cases} (e^{4\pi\sqrt{-1}t}, 1) & \text{if } 0 \le t \le \frac{1}{2} \\ (1, e^{4\pi\sqrt{-1}(t-\frac{1}{2})}) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

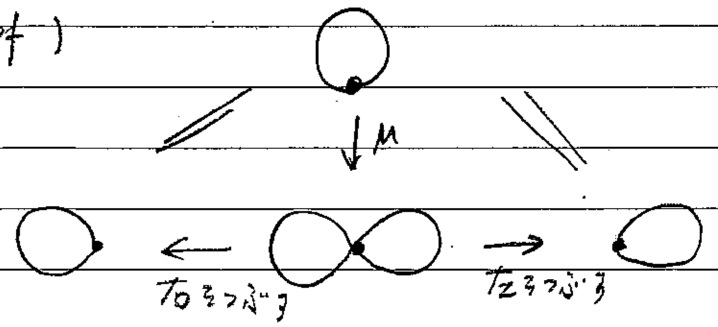
$* : S^1 \to S^1 \quad {}^\forall z \mapsto 1$

補題4.6.

$$
\begin{tikzcd}
 & S^1 \arrow[dl, "1"'] \arrow[d, "\mu"] \arrow[dr, "1"] & \\
S^1 & S^1 \vee S^1 \arrow[l, "1\vee *"] \arrow[r, "*\vee 1"'] & S^1
\end{tikzcd}
$$

基点を動かさないhomotopyを除いて可換

(略)



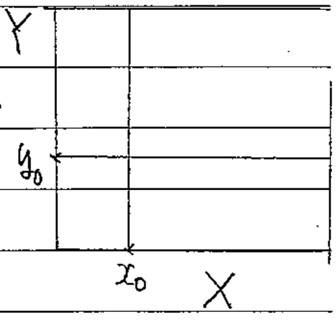
$((S^1, 1), \mu)$ co H space) という (定義はプリント)
μ: coproduct

wedge 和のhomology

(X, x_0) : pointed space.

条件(4.5) $x_0 \in {}^{\exists}U_0 \overset{\text{open}}{\subset} X$ $\{x_0\} \subset U_0$ 変位retract

(Y, y_0) : (4.5)をみたす pointed space



$X \vee Y := \{(x, y) \in X \times Y : x = x_0$ または $y = y_0\}$
wedge sum

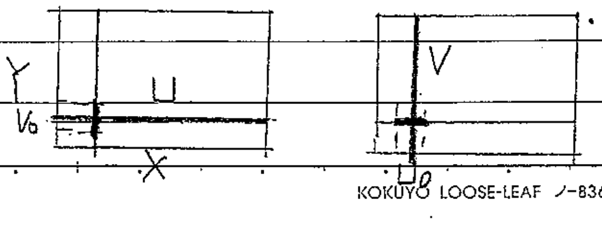
$i_1 : X \to X \vee Y$ $x \mapsto (x, y_0)$ 中への同相
$i_2 : Y \to X \vee Y$ $y \mapsto (x_0, y)$

補題4.8 このとき $\forall q \geq 1$

$$i_{1*} + i_{2*} : H_q(X) \oplus H_q(Y) \xrightarrow{\cong} H_q(X \vee Y)$$

(証) $y_0 \in V_0 \overset{\text{open}}{\subset} Y$ $\{y_0\} \subset V_0$ 変位retract

$U = i_1(X) \cup i_2(V_0)$
$V = i_1(U_0) \cup i_2(Y)$



$\{U, V\}$: open covering of $X \vee Y$

$i_1 : X \xrightarrow{\simeq} U$
$i_2 : Y \xrightarrow{\simeq} V$) homotopy 同値

$U \cap V \simeq *$

Mayer-Vietoris exact sequence

$$\begin{tikzcd} H_q(U\cap V) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_q(U)\oplus H_q(V) \arrow[r, "\cong"] & H_q(X\vee Y) \arrow[r] & \tilde{H}_{q-1}(U\cap V) \arrow[d, phantom, "\shortparallel"] \\ H_q(*) \arrow[d, phantom, "\shortparallel"] & H_q(X)\oplus H_q(Y) \arrow[u, "\cong", "i_{1*}+i_{2*}"'] & & \tilde{H}_{q-1}(*) \arrow[d, phantom, "\shortparallel"] \\ 0 & & & 0 \end{tikzcd}$$

//

系4.9 $\forall u \in H_1(S^1)$

$$\mu_*(u) = i_{1*}u + i_{2*}u \in H_1(S^1)$$

(証) $\exists u_1, \exists u_2 \in H_1(S^1)$ $\mu_*(u) = i_{1*}u_1 + i_{2*}u_2$

両辺に $(1_{S^1} \vee *)_*$ を施す（右側をつぶす）

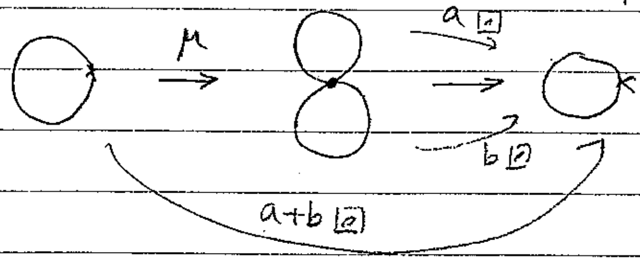
$u = u_1$

左側をつぶすと $u = u_2$ //

(pf of Th4.5)

主張(4.6) $\forall a, \forall b \in \mathbb{Z}$, $(\varphi_a \vee \varphi_b)\circ\mu \simeq \varphi_{a+b} : S^1 \to S^1$

(pf)



//

$\forall u \in H_1(S^1)$

$$(\varphi_{a+b})_*(u) = (\varphi_a \vee \varphi_b)_*(\mu_* u) = (\varphi_a \vee \varphi_b)_*(i_{1*}u + i_{2*}u)$$
$$= \varphi_{a*}(u) + \varphi_{b*}(u)$$

ゆえに
$$\begin{cases} \deg \varphi_{a+b} = \deg \varphi_a + \deg \varphi_b \\ \deg \varphi_1 = \deg(1_{S^1}) = 1 \end{cases}$$ 準同型

$\Rightarrow \deg \varphi_a = a$ // Th.4.5

$\mathbb{C}P^1$ についての計算

$\mathbb{C}P^1 = \{[z:w] \in \mathbb{C}P^1 ;\ (z,w) \neq (0,0)\}$

$\cong S^2$, $\ni \infty := [0:1]$

同一視 $\mathbb{C} \Rightarrow \mathbb{C}P^1 \quad z \mapsto [1:z]$

$m \geqq 1$

$f(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m$ 多項式

$(a_0, a_1, \cdots, a_m \in \mathbb{C},\ a_0 \neq 0)$

$f : \mathbb{C}P^1 \to \mathbb{C}P^1$ 連続写像

$z \mapsto f(z)$,

$\infty \mapsto \infty$

定理4.10 $\deg(f) = m$

(証) $e^\alpha = a_0$ なる $\alpha \in \mathbb{C}$ をとる

$F : \mathbb{C}P^1 \times [0,1] \to \mathbb{C}P^1$ 連続

$(z,t) \longmapsto e^{\alpha t} z^m + t\{a_1 z^{m-1} + \cdots + a_m\}$

$F(z,1) = f(z) \quad F(z,0) = z^m =: g(z)$ とおく

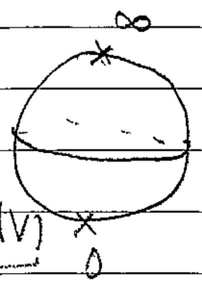
$\deg(f) = \deg(g) = \deg(z^m)$

示すべきこと
$\deg g = m$

$U := \mathbb{C},\ V := \mathbb{C}P^1 \setminus \{0\} \qquad U \simeq V \simeq *$

$\{U, V\}$: open covering of $\mathbb{C}P^1$

$$\underbrace{H_2(U) \oplus H_2(V)}_{=0} \to H_2(\mathbb{C}P^1) \xrightarrow[\cong]{\partial_*} H_1(U \cap V) \to \underbrace{H_1(U) \oplus H_1(V)}_{=0}$$



$i: S^1 \hookrightarrow U \cap V = \mathbb{C} \setminus \{0\}$ 変位retract

$g(U) \subset U,\ g(V) \subset V,\ g(S^1) \subset S^1$

$g|S^1 = \varphi_m$

∂_* の自然性から

$$\begin{array}{ccccc} H_2(\mathbb{C}P^1) & \xrightarrow[\cong]{\partial_*} & H_1(U \cap V) & \xleftarrow[\cong]{i_*} & H_1(S^1) \\ g_* \downarrow & \circlearrowleft & g_* \downarrow & \circlearrowleft & {\varphi_m}_* \downarrow \\ H_2(\mathbb{C}P^1) & \xrightarrow[\cong]{\partial_*} & H_1(U \cap V) & \xleftarrow[\cong]{i_*} & H_1(S^1) \end{array}$$

$\deg \varphi_m = m.$

ゆえに $\deg g = m$ // Th 4.10

系 4.11 (代数学の基本定理)

$m \geq 1$

$f(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m$

$(a_0, a_1, \cdots, a_m \in \mathbb{C},\ a_0 \neq 0)$

$\Longrightarrow f(z)$ は 根 $\in \mathbb{C}$ をもつ

(証) $f: \mathbb{C}P^1 \to \mathbb{C}P^1$

$\deg(f) = m \neq 0$ より f: 全射

$\exists z \in \mathbb{C}P^1$ $f(z) = 0$

$f(\infty) = \infty$ より $z \in \mathbb{C}$ //

注意 $\deg(z^{-1}) = 1 \neq -1$