§13 多様体の基本類

$n \geq 1$ fixed

今日の目標

Theorem 11.13 X : n-dim oriented closed (compact かつ $\partial X = \phi$) connected C^∞ manifold

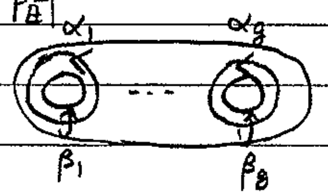
$\Rightarrow H_n(X) \cong \mathbb{Z}$

$\exists! [X] \in H_n(X)$ 生成元 $\forall p \in X \quad j_{p*}[X] = \mu_p \in H_n(X, X \setminus \{p\})$

(ただし j_{p*} : inclusion homom.
μ_p : 向きからきまる生成元)

とくに X は条件 (11.0) をみたす

のこりの時間

$H_1($  $) = \bigoplus_{i=1}^{g} (\mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i) \cong \mathbb{Z}^{2g}$

$$H_*(\qquad\qquad) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2 \\ 0 & \text{if } * \geq 3 \end{cases}$$

Def X : topological n-manifold

$\overset{\text{def}}{\Longleftrightarrow}$ 1) X : Hausdorff space

2) $\forall p \in X \quad p \in \exists U \overset{\text{open}}{\subset} X \quad \exists V \overset{\text{open}}{\subset} \mathbb{R}^n$

$\exists \varphi : U \xrightarrow{\approx} V$ homeomorphism

(U, φ, V) chart

記号 $x \in \mathbb{R}^n, r > 0$

$B_n(x, r) := \{ y \in \mathbb{R}^n ; \| y - x \| \leq r \}$

$\mathring{B}_n(x, r) := \{ \qquad\qquad \lneq r \}$

(但 $\| (y_1, \cdots, y_n) \| = (\Sigma y_i^2)^{1/2}$ Euclid norm)

No. Date

考え方

X : C^∞ mfd $\qquad U \overset{\text{open}}{\subset} \mathbb{R}^n$

$$H^*(X;\mathbb{R}) \underset{\text{de Rham Theorem}}{\cong} H^*_{DR}(X) \overset{\text{対応}}{\longleftrightarrow} \mathcal{E}(U) = C^\infty(U;\mathbb{R})$$

$\updownarrow$ dual $\qquad\qquad \updownarrow$ dual

$$H_*(X;\mathbb{R}) \overset{\text{対応}}{\longleftrightarrow} \mathcal{E}'(U) = \text{distribution with compact support}$$

$\Leftarrow$ compact 台をもつ distribution

补題 13.1. X : locally compact Hausdorff space.

$\forall q \geq 0$ $\forall u \in H_q(X)$, $\exists U \overset{\text{open}}{\subset} X$ relatively compact ($\bar{U}$: compact)

$u \in \underset{\text{inclusion homom.}}{\mathrm{Im}\,(H_q(U) \to H_q(X))}$

(証) $\exists z \in Z_q(S_*(X)) \subset S_q(X)$, $u = [z] \in H_q(X)$

$z = \sum_{i=1}^m a_i \sigma_i$, $a_i \in \mathbb{Z}$, $\sigma_i : \Delta^q \to X$ conti

$\bigcup_{i=1}^m \sigma_i(\Delta^q)$: compact

$\subset \exists U \overset{\text{open}}{\subset} X$ relatively compact ($\Leftarrow X$: loc. cpt Hausdorff) //

X : top. n-mfd.

$K \subset L \subset X$ compact subsets

$X - L \subset X - K$

$j_{K*} = j_{K*}^L : H_q(X, X-L) \to H_q(X, X-K)$ inclusion homom.

L 上の「函数」の K への制限

$K = \{p\}$ のとき

$j_{p*} = j_{\{p\}*} : H_q(X, X-L) \to H_q(X, X-\{p\})$

と表す。

<u>補題 13.2</u> X : top n-mfd, $K \subset X$ compact subset

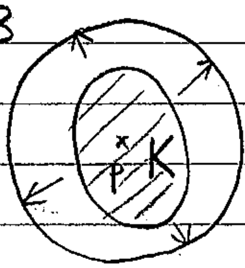
(1) $\forall q \neq n \quad H_q(X, X-K) = 0$

(2) $u \in H_n(X, X-K)$, $\forall p \in K \quad j_{p*}u = 0$

$\Longrightarrow u = 0$

(証) <u>Step 1</u> $X = \mathbb{R}^n$, K : compact 凸 の場合

$p \in K \quad \exists R > 0, \quad K \subset B_n(p, R) =: B$



$H_*(X, X-K) \cong H_*(X, X-B)$

$\cong H_*(X, X-\{p\}) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{if } * \neq n \end{cases}$

<u>Step 2</u> $K_1, K_2, K_1 \cap K_2$ について Lem が成立つ

$\Longrightarrow K_1 \cup K_2$ についても Lem が成立つ

(証) $(X-K_1) \cup (X-K_2) = X - (K_1 \cap K_2)$

$(X-K_1) \cap (X-K_2) = X - (K_1 \cup K_2)$

対の Mayer-Vietoris 完全列 (Th 10.9 (4))

$$H_{q+1}(X, X-(K_1 \cap K_2)) \to H_q(X, X-(K_1 \cup K_2)) \xrightarrow{(j_{K_1*}, -j_{K_2*})} H_q(X, X-K_1) \oplus H_q(X, X-K_2)$$

$q \neq n$ のとき

$$0 \to H_q(X, X-(K_1 \cup K_2)) \to 0$$

$q = n$ のとき

$$0 \to H_n(X, X-(K_1 \cup K_2)) \to H_n(X, X-K_1) \oplus H_n(X, X-K_2)$$

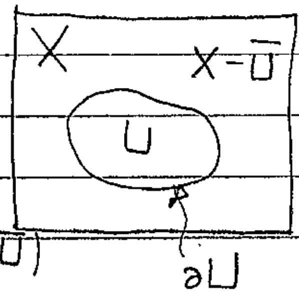
$$\begin{array}{ccc} \cup\!\!\!\!\!\!\;\; & & \cup\!\!\!\!\!\!\;\; \\ u & \longmapsto & (j_{K_1*}u, \ -j_{K_2*}u) \\ & \forall p & \downarrow \qquad\qquad \downarrow \\ & & j_{p*}u = 0 \qquad -j_{p*}u = 0 \end{array}$$ //

Step 1 と Step 2 によってはりあわせていく。 //

No. Date . .

定理 13.3 X : top n-mfd
$\Longrightarrow \forall q \neq n \quad H_q(X) = 0$

(証) $\forall U \overset{\text{open}}{\subset} X$ rel. compact. $H_q(U) = 0$
を示せばよい



$(X, X-\partial U, X-\bar{U})$ の homology exact seq.

$$\begin{array}{ccccc} H_{q+1}(X, X-\partial U) & \to & H_q(X-\partial U, X-\bar{U}) & \to & H_q(X, X-\bar{U}) \\ \| & & \|\wr\, \text{exc} & & \| \\ 0 & & H_q(U) & & 0 \end{array}$$

//

補題 13.4. X : top n-mfd
$u \in H_n(X)$
$S_u := \{ p \in X : j_{p*} u = 0 \} \subset X$ open かつ closed.

(証) $p \in \mathring{B} \subset X$ ball B をとる

$$H_n(X, X-\{p\}) \xleftarrow{\cong} H_n(X, X-B) \xrightarrow{\cong} H_n(X, X-\{p'\}) \quad \forall p' \in \mathring{B}$$

$$\begin{array}{ccccc} j_{p*} u = 0 & \Rightarrow \forall p' \in \mathring{B} & j_{p'*} u = 0 & \Rightarrow & \mathring{B} \subset S_u \\ \neq 0 & & \neq 0 & & \mathring{B} \subset X - S_u \end{array}$$

ゆえに S_u, $X - S_u$ ともに open //

定理 13.5. X : conn. non-compact. n-mfd
$\Longrightarrow H_n(X) = 0$.

(証) $\forall u \in H_n(X)$, $\exists p \in X$ \ ("support" of u)
$S_u = \{ p' \in X : j_{p'*} u = 0 \} \neq \phi$, 開かつ閉 $\overset{\text{ゆえに}}{=} X$
Lem 13.2 より $u = 0$ //

基本類の構成

$\mu_0 \in H_n(\mathbb{R}^n, \mathbb{R}^n-\{0\}) \cong \mathbb{Z}$ 生成元 1つ固定する

$^\forall x \in \mathbb{R}^n,\ ^\exists R>0\quad x \in B_n(0,R) =: B$

$$\begin{array}{ccccc} H_n(\mathbb{R}^n, \mathbb{R}^n-\{0\}) & \cong & H_n(\mathbb{R}^n, \mathbb{R}^n-B) & \cong & H_n(\mathbb{R}^n, \mathbb{R}^n-\{x\}) \\ \psi & & \psi & & \psi \\ \mu_0 & \longmapsto & \mu_B & \longmapsto & \mu_x \end{array}$$

μ_B と書く　μ_x が定まる.

X : n-dim. oriented C^∞ mfd

$^\forall p \in X,\ ^\exists (U, \varphi, V)$: 正のchart. $p \in U$

$$\mu_p := \varphi^{-1}_* \mu_{\varphi(p)} \in H_n(X, X-\{p\})\ \text{生成元}$$

正のchartのとり方によらない

定理 13.6 X : oriented C^∞ n-manifold

$K \subset X$ compact

$\Rightarrow\ ^{\exists!}\mu_K \in H_n(X, X-K)$ s.t. $^\forall p \in K$

$$j_{p*}\mu_K = \mu_p \in H_n(X, X-\{p\})$$

(証) 一意性 $\Leftarrow$ Lem 13.2

存在 Step. 1 $K_1, K_2 \subset X$ compact.

μ_{K_1}, μ_{K_2} が存在する.

$\Longrightarrow \mu_{K_1 \cup K_2}$ が存在する.

(証) Mayer-Vietoris 完全列

$$\begin{array}{ccc} H_n(X, X-(K_1\cup K_2)) & \xrightarrow{(j_{K_1*},\,-j_{K_2*})} & H_n(X, X-K_1) \oplus H_n(X, X-K_2) \\ & & \downarrow j_{K_1\cap K_2*} \\ & & H_n(X, X-(K_1\cap K_2)) \end{array}$$

$^\exists \mu_{K_1\cup K_2} \longmapsto (\mu_{K_1}, -\mu_{K_2}) \mapsto 0$

$\left(\begin{array}{l} \because\ ^\forall p \in K_1 \cap K_2 \\ j_{p*}(\mu_{K_1} - \mu_{K_2}) = \mu_p - \mu_p = 0 \\ \text{Lem 13.2}\ // \end{array}\right)$

//

Step. 2 $X = \mathbb{R}^n$ の場合

$\exists R > 0 \quad K \subset B_n(0,R) \ (=: B)$

$\mu_K := j_{K*}\mu_B$ とおけ. —

あとは Step 1 を使って Step 2 をはりあわせる. //

定理 11.13 X : n-dim oriented closed connected C^∞ mfd

$\Longrightarrow H_n(X) \cong \mathbb{Z}$

$\exists! [X] \in H_n(X) \quad \forall p \in X \quad j_{p*}[X] = \mu_p$

(証) $\mu_X =: [X]$ とおけ. (Th. 13.6)

$\forall p \in X$ について $j_{p*} : H_n(X) \to H_n(X, X - \{p\}) (\cong \mathbb{Z})$ が

同型であることを示せばよい

(全射) $j_{p*}\mu_X = \mu_p$ より明らか

(単射) $u \in \mathrm{Ker}(j_{p*} : H_n(X) \to H_n(X, X - \{p\}))$ とする

$S_u := \{p' \in X \ ; \ j_{p'*}u = 0\} \subset X$ 開かつ閉

$\quad\ \ \rotatebox{90}{$\in$}$ $\qquad \ne \phi \quad X$: conn. ゆえに $X = S_u$.

$\quad\ \ p$

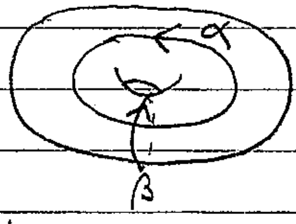
Lem 13.2 より $u = 0$ //

以下 $[S^1] \in H_1(S^1) \cong \mathbb{Z}$ を fix する.

$l : S^1 \to X$ conti. map について

$l := l_*[S^1] \in H_1(X)$ とかくことにする.

T^2 の homology と $GL_2(\mathbb{Z})$ の作用

$T^2 = S^1 \times S^1 =$ 

closed orientable conn. 2-mfd

$S^1 < \mathbb{C}^\times$

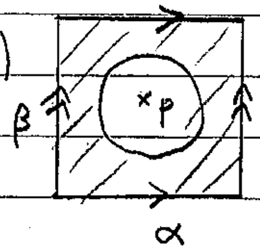
$S^1 \to S^1 \times S^1$

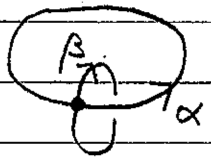
$\alpha : z \longmapsto (z, 1)$

$\beta : w \longmapsto (1, w)$

$$H_s(T^2) = \begin{cases} \mathbb{Z} & \text{if } s=0,2 \\ 0 & \text{if } s \geqq 3 \end{cases}$$

<u>定理13.7</u> $H_1(T^2) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \cong \mathbb{Z}^2$

(証)  $= O\ (= T')$

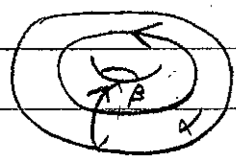
$O \simeq$  $\approx S^1 \vee S^1$

$$H_*(T^2, O) \underset{exc.}{\cong} H_*(I^2, \partial I^2) = \begin{cases} \mathbb{Z} & \text{if } *=2 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{tikzcd} H_2(T^2) \arrow[r] \arrow[dr, Rightarrow] & H_2(T^2, O) \arrow[r, "0"] \arrow[d, "\cong"] & H_1(O) \arrow[r, "\cong"] \arrow[d, equal] & H_1(T^2) \arrow[r] & H_1(T^2, O) \arrow[d, equal] \\ & H_2(T^2, T^2 \setminus \{p\}) & \mathbb{Z}\alpha \oplus \mathbb{Z}\beta & & 0 \end{tikzcd}$$

//

$p_i : T^2 = S^1 \times S^1 \to S^1$ 第i射影 $i=1,2$

$\begin{cases} p_{1*}\alpha = [S^1] \\ p_{1*}\beta = 0 \end{cases}$ $\begin{cases} p_{2*}\alpha = 0 \\ p_{2*}\beta = [S^1] \end{cases}$ 

ゆえに

$$(p_{1*}, p_{2*}) : H_1(T^2) \xrightarrow{\cong} H_1(S^1) \oplus H_1(S^1)$$
$$\alpha \longmapsto (1, 0)$$
$$\beta \longmapsto (0, 1)$$

$$GL_2(\mathbb{Z}) \curvearrowright T^2 = S^1 \times S^1 \subset \mathbb{C}^\times \times \mathbb{C}^\times$$

$\rotatebox{90}{$\in$}\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\rotatebox{90}{$\in$}\ (z, w)$

$$A(z, w) := (z^a w^b, z^c w^d)$$

または $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ とみなして

$$A\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \bmod \mathbb{Z}^2\right) := \begin{pmatrix} at_1 + bt_2 \\ ct_1 + dt_2 \end{pmatrix} \bmod \mathbb{Z}^2 \quad \left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{Z}^2\right)$$

定理 13.8. (1) $A_*[T^2] = (\det A)[T^2]$

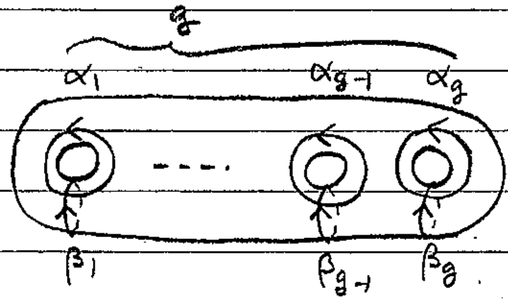
(2) $A_*(x\alpha + y\beta) = (ax+by)\alpha + (cx+dy)\beta \quad (x, y \in \mathbb{Z})$

(証) (1) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \bmod \mathbb{Z}^2$ において Jacobian を計算する

(2) $\left.\begin{aligned} p_1((A\circ\alpha)(z)) &= z^a \\ p_2((A\circ\alpha)(z)) &= z^c \end{aligned}\right\} \Rightarrow A_*\alpha = a\alpha + c\beta$

$\left.\begin{aligned} p_1((A\circ\beta)(w)) &= w^b \\ p_2((A\circ\beta)(w)) &= w^d \end{aligned}\right\} \Rightarrow A_*\beta = b\alpha + d\beta$ //

$g \geq 0$

$\Sigma_g =$ 

closed oriented connected 2-mfd

$\Sigma_0 = S^2, \quad \Sigma_1 = T^2.$

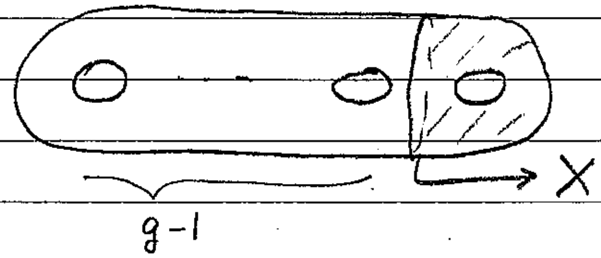
$$H_s(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } s = 0, 2 \\ 0 & \text{if } s \geq 3 \end{cases}$$

定理 13.9 $H_1(\Sigma_g) = \bigoplus_{i=1}^{g} (\mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i) \cong \mathbb{Z}^{2g}$

証明 g についての帰納法

$g = 0, \quad \Sigma_0 = S^2, \quad H_1(S^2) = 0$ 明らか

$g \geqq 1$ とする. $g-1$ まで示されているとする



$X =$  $\simeq S^1 \vee S^1$ (Lem 13.7の)

$$H_*(\Sigma_g, X) = \tilde{H}_*(\Sigma_g / X) = \tilde{H}_*(\Sigma_{g-1})$$

$$\begin{tikzcd}
H_2(\Sigma_g) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & H_2(\Sigma_g, X) \arrow[r] \arrow[dl, "\cong"] & H_1(X) \arrow[r] \arrow[d, equal] & H_1(\Sigma_g) \arrow[r] & H_1(\Sigma_g, X) \arrow[r] \arrow[d, equal] & \tilde{H}_0(X) \arrow[d, equal] \\
H_2(\Sigma_g, \Sigma_g \setminus \{p\}) & & \mathbb{Z}\alpha_g \oplus \mathbb{Z}\beta_g & & \bigoplus_{i=1}^{g-1} (\mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i) & 0
\end{tikzcd}$$

$p \in \Sigma_g \setminus X$

$\mathbb{Z}$-free

帰納法が進む //

lens space. $L(n;k) = S^3 / (\mathbb{Z}/n)$

$$S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$$

$$(j \bmod n)(z,w) = (\zeta^j z, \zeta^{jk} w), \quad \zeta = e^{2\pi\sqrt{-1}/n}$$

定理 13.10 $$H_q(L(n,k)) = \begin{cases} \mathbb{Z} & \text{if } q = 0, 3 \\ 0 & \text{if } q = 2 \\ \mathbb{Z}/n & \text{if } q = 1 \\ 0 & \text{if } q \geqq 4 \end{cases}$$

分解 $L(n,k) = S^1 \times D^2 \cup D^2 \times S^1$ を使う //