Global Focus on Knowledge -Creating Mathematics-

Lecture three

Mathematics "On Campus"

Creating words, Originating worlds

2009.10.22

Number System

- Natural number 0, 1, 2, 3, 4, ….
- Integer 0, $\pm 1, \pm 2, \pm 3, \pm 4$, ….
- Rational number $\frac{4}{3}, -\frac{7}{8}$, 0, ….
- Real number $\sqrt{2}$, e = 2.7182818….,

  π=3.141592653…., ….
- Complex number $\sqrt{-1}, \frac{-1 \pm \sqrt{-3}}{2}$,

  $\cos\frac{2\pi}{7} + \sqrt{-1}\sin\frac{2\pi}{7}$, ….

at elementary school

- Numerical position

  .... 1st -4th grade at elementary school
- 0 (any number when multiplied by zero is zero) .... 3rd grade
- fraction (Integer to rational number).... 4th grade
- decimal (rational number to real number).... 4th grade
- Number line (real number) .... 4th grade

at and after Junior high school

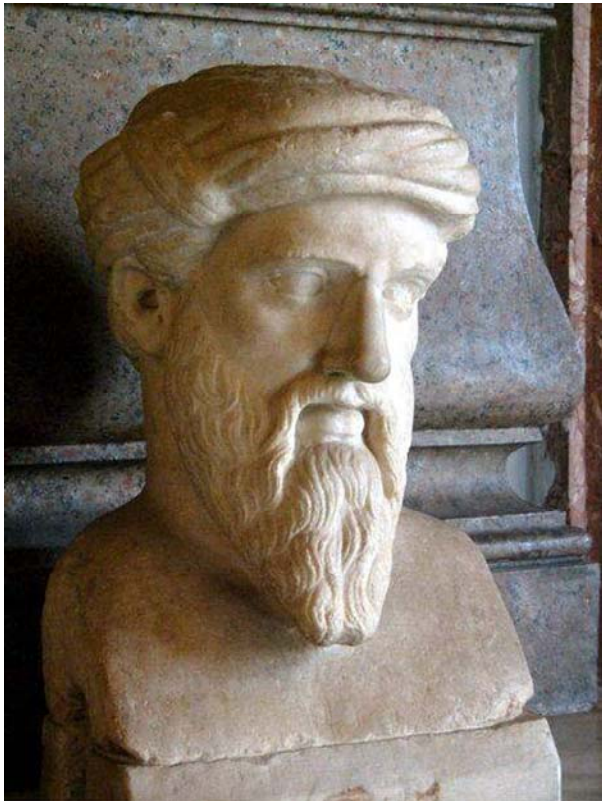
- Negative number(natural number to Integer).... 1st grade at junior high school
- Square root(rational number to real number).... 3rd grade at junior high school
- Prime number(Integer) .... 3rd grade at junior high school
- $\sqrt{2}$ is not a rational number .... 1st grade at high school
- Mathematical induction(natural number)....
  2nd grade at high school
- Imaginary number (real number to complex number)....
  2nd grade at high school
- the definition of real number .... 1st grade at university?

In the historical order

- Fraction .... Ancient Mesopotamia
- $\sqrt{2}$ is an irrational number....Ancient Greek

  Pythagorean school (6c. BC)
- The theory of ratio....real number theory at Ancient Greek

  Eudoxos (4c. BC)
- Numerical position, 0 .... India(around 6c)
- Negative number .... India(around 7c)



Pythagoras

(580 BC? - 497 BC?)

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In the historical order

- decimal …. end of 16c.
- Imaginary number …. Cardano(1545)
- Fundamental theorem of algebra
  …. Gauss(1799)

In the mathematical order

- natural number to integer .... subtraction
- integer to rational number.... division
- rational number to real number .... length, area

  differentiation/integration
- real number to complex number ....

  solution of equation

real number to complex number

- the Cubic Formula

  (Cardano formula)

  the concept of imaginary number is necessary to get to the solution, even if the solutions are real.

Cardano (1501.9.24 - 1576.9.21)

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Cardano's formula (1535,1545)

- Cubic equation $X^3 = 3pX + q$

subsidiary「quadratic」equation

$$Y^3 + Z^3 = q,\ Y Z = p$$

$$X = Y + Z,\ \omega Y + \omega^2 Z,$$

$$\omega^2 Y + \omega Z.\ (\omega^3 = 1)$$

Cardano's formula (1535,1545)

- $X^3 = 15X + 4 = 3pX + q$

$Y^3 + Z^3 = q = 4, \; Y Z = p = 5$

$Y^3, Z^3 = 2 \pm 11\, i$

$X = Y + Z,$

$= (2 + i) + (2 - i) = 4$

Real number to Imaginary number

- Fundamental theorem of algebra

  (d'Alembert/Gauss theorem)

All equations have complex number solution(s)

No need to extend the world of number system(the end of story?)

Gauss (1777.4.30 - 1855.2.23)

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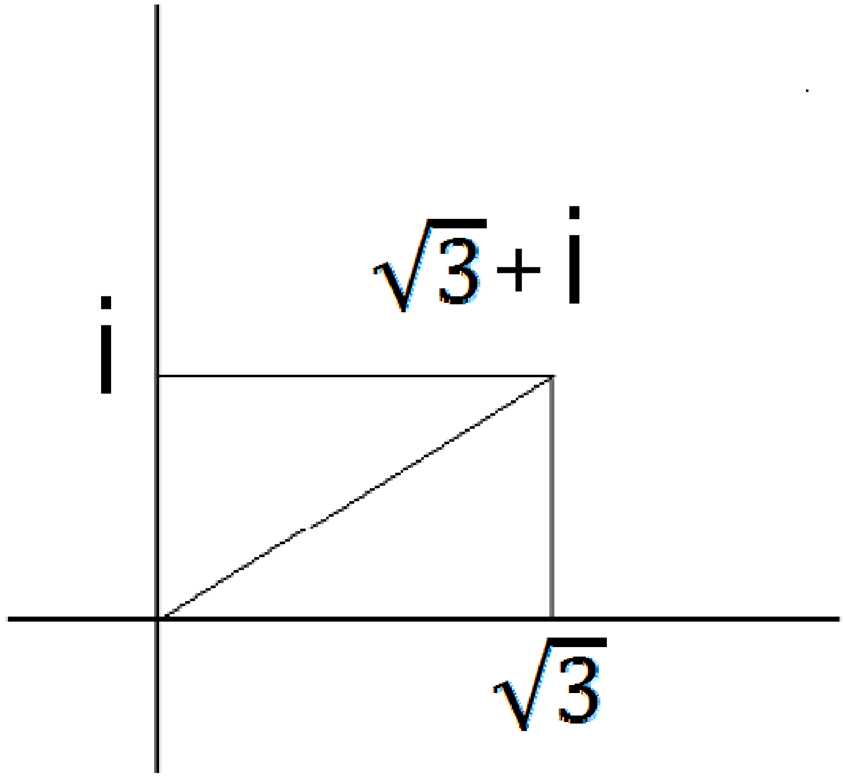
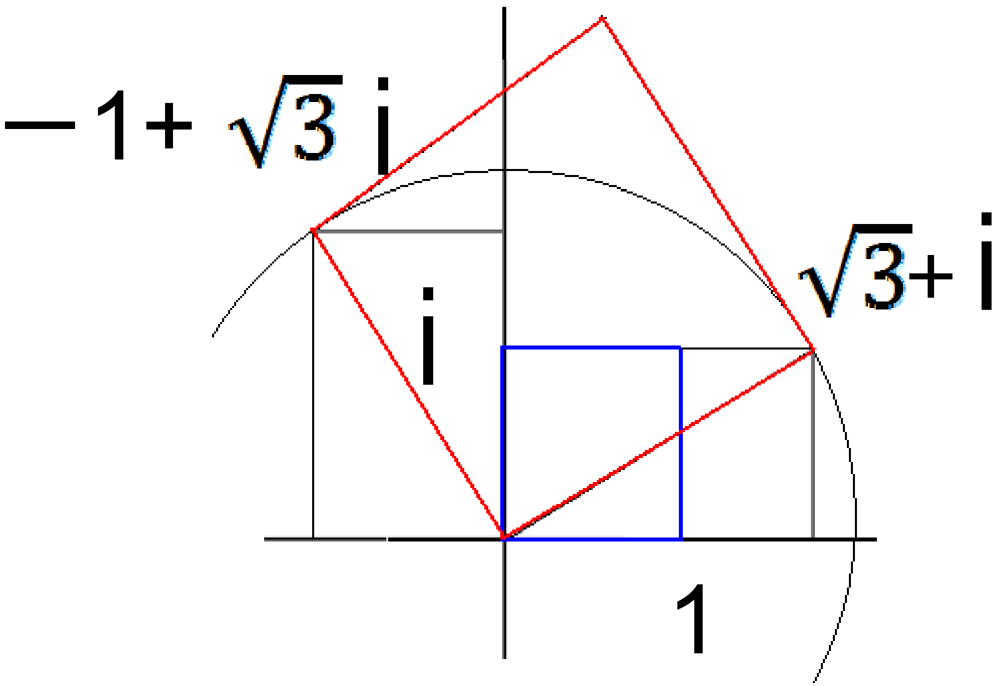
Complex plane (Gauss plane)

- A complex number=a point on the plane



A complex number=a point on the plane

- multiplication of complex numbers : rotation and extension



Complex numbers and matrices

multiplied by $\sqrt{3}+i$: 1 to $\sqrt{3}+i$

i to $-1+\sqrt{3}\,i$

multiplied by $\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$

Complex numbers to Quaternions

- Complex number : $a + b\,i$ (a, b are real)
- quaternion : $a + b\,i + c\,j + d\,k$

  (a, b, c, d are real)

  (Hamilton 1843.10.14)
- $i^2 = j^2 = k^2 = -1,\ \ i\,j = -\,j\,i = k$
- Giving up the commutatibility

Hamilton (1805.8.4 –1865.9.2)

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In the latter half of the 19th century

- the definition of Natural number....

  Peano axioms (1891)

- 0 is a natural number.
- if n is a natural number, then n + 1 is also a natural number
- there is no natural number n such that 0 = n + 1
- if n + 1 = m + 1 then n = m
- The principle of mathematical induction

Peano (1858.8.27 – 1932.4.20)

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In the latter half of 19^{th} century

- the definition of real number ....

  Cantor the founder of set theory (1872)

  Dedekind (1858.11.24)

  foundations of the world of number system established

  (the end of story this time?)

Cantor(1845.3.3 –1918.1.6)

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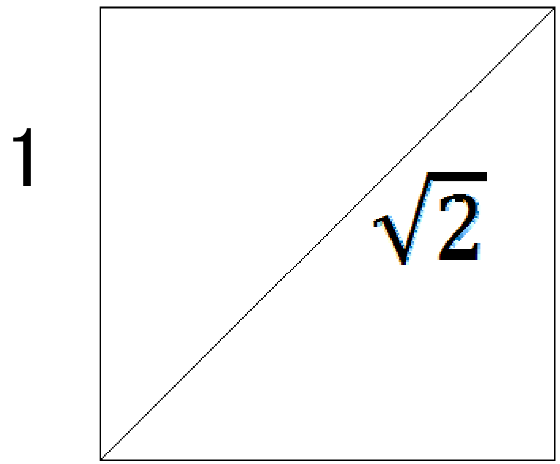
mathematically

- natural number to integer…subtraction
- Integer to rational number…division
- Rational number to real number…length,area

  differentiation/integration
- real number to complex number....
  solution of equation

Rational number to real number....length

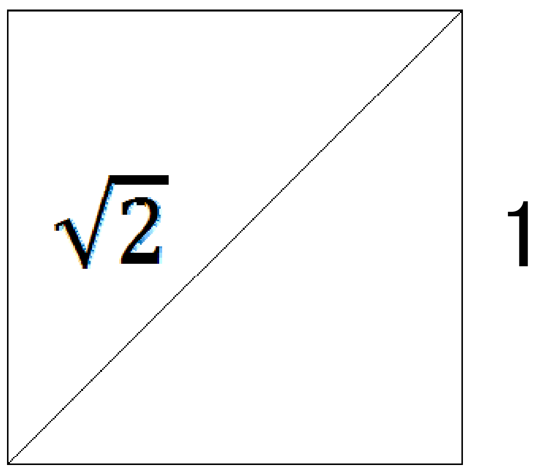
The ratio between the length of the side of a square and the diagonal $\sqrt{2}$ is



not a rational number

有理数から実数へ・・・長さ

正方形の１辺と対角線の長さの比$\sqrt{2}$は



有理数ではない

$\sqrt{2}$ is not a rational number

suppose

$$\sqrt{2} = \frac{n}{m} \text{ (irreducible) then}$$

$2m^2 = n^2$ $\Rightarrow$ n is even $n=2n'$

$m^2 = 2n'^2$ $\Rightarrow$ m is even

$\Rightarrow$ contradiction

rational number to real number....area

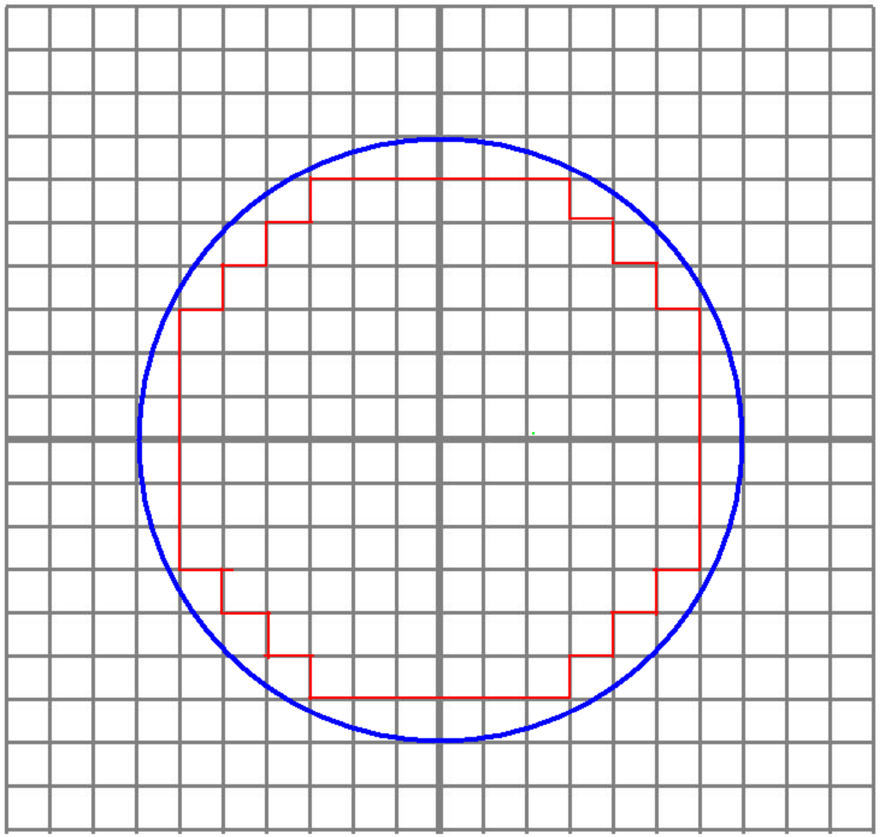
- The area of a disk
- The number of □

  times

  the area of each □

The small □ limit

The area of a disk



Rational number to real number··· differentiation, integration, limit

- Exponential function $(e^x)' = e^x$, $e^0=1$,

$e = 2.7182818....$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$$

$$= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

What is real number?

- A point on a line
  - what is a point on a line?....

real number

a circular argument

- the limit of rational number
- A real number can be expressed as (infinite)decimal

斎藤　毅

すべての道は素数へ続く

3世紀以上もの間、数多くの数学者を退け続けてきたフェルマーの最終定理がついに証明されたことが話題になったことを覚えている方も多いだろう。これが解決されたからといって、整数に関する問題がすべて明らかになったということではない。しかしこの成果は、人間がこれまでに築きあげてきた、数に対する理解の1つの頂点を示すものである。それは同時に、整数のこれからの発展への展望を切りひらくものでもある。

フェルマーの最終定理そのものは中学生にも説明できるといわれるように、数論では、誰もが知っている数の性質を探求する。しかし、問題が素朴だから簡単だというわけではない。素朴にみえる問題の難しさは、手がかりが見えないところにある。19世紀の数学界に君臨したガウスは、数論を「数学の女王」と呼んだ。それは、数論それ自体の魅力とともに、数学を1つに結びつける数論のもつ力をさすものだろう。数学の1つの分野での発見は、数に対する理解への隠されていた手がかりとして、数論の進展につながるのである。

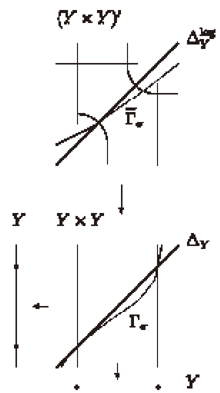
人類の数に対する理解は、数の世界を広げることで進んできた。1つ、2つと数えることではじまる自然数から、引き算で整数へ、割り算で分数へという具合である。それに対し、ふつうの数と思われている実数は、数直線上の点としてとらえられる。これがあまりに自然な考え方だったため、そのほんとうの意味が明らかにされたのは、19世紀も終わりに近づいてからだった。

このように数の世界を広げる方法が認識されると直ちに、同じようにして数の世界を広げる方法がいくらでもあることが発見された。2, 3, 5, 7、…と、素数がいくらでもあることは、古代ギリシャのユークリッド「原論」で証明されていることだが、その素数1つ1つに、2進数の世界、3進数の世界、…という数の体系が見いだされた。フェルマーの最終定理が20世紀に証明できたのは、整数を2進数、3進数、…と考えることが、整数を実数と考えるのと同じくらい自然なことであり、重要なことであるということがわかったからだ、ということができる。

19世紀の解析学の大きな成果である関数論は、素数の分布の研究に応用され、現代数学の最大の未解決問題とよばれるリーマン予想を生み出した。20世紀の抽象数学の花形である代数幾何は、ヴェイユ予想の解決をもたらし、素数を「点」と考える視点が確立した。群の無限次元表現論は、類体論の非可換化を可能にし、現代の数論の中心的問題であるラングランヅ対応へと導いた。フェルマーの最終定理の素朴さの影に隠されていた手がかりは、このラングランヅ対応だった。

この、代数幾何を使って整数論を研究する数論幾何が、私の研究分野である。これは、p進体、代数幾何、ガロワ表現が交錯する活発な研究領域である。数のもつ隠された対称性は、ガロワ群の中に読み取ることができる。ガロワ群と代数幾何は、ヴェイユ予想の証明を可能にしたエタール・コホモロジーによって結びついている。この結びつきを通して、両者の関わりを調べる、幾何的な分岐理論が専門である。ラングランヅ対応と分岐理論の関係についての研究は、現在世界的に活発に研究が進んでいるp進局所ラングランヅ対応への契機を与えるものとなった。

数論と微分方程式は、一見かけ離れているように見えるが、グロタンディークらによるエタール層の理論と、佐藤幹夫、柏原正樹らによって創始された微分方程式のD加群の理論の間には、表面的なものにとどまらない密接な類似がある。この数年は、超局所解析と分岐理論の類似を研究し、特性多様体の定義やオイラー数の公式などの成果があがっている。



creating the system of number

- The human understanding of numbers has made progress as human extended the world of number.
- It begins with natural numbers,

  as we count things, one, two, …

  then subtraction leads to the concept of integer,

  division to fraction, and so forth.
- On the contrary, real number which is thought to be ordinary number, is considered to be a point on the number line.
- This way of thinking was so natural that it was not until the latter 19th century that the true property of real numbers was understood.

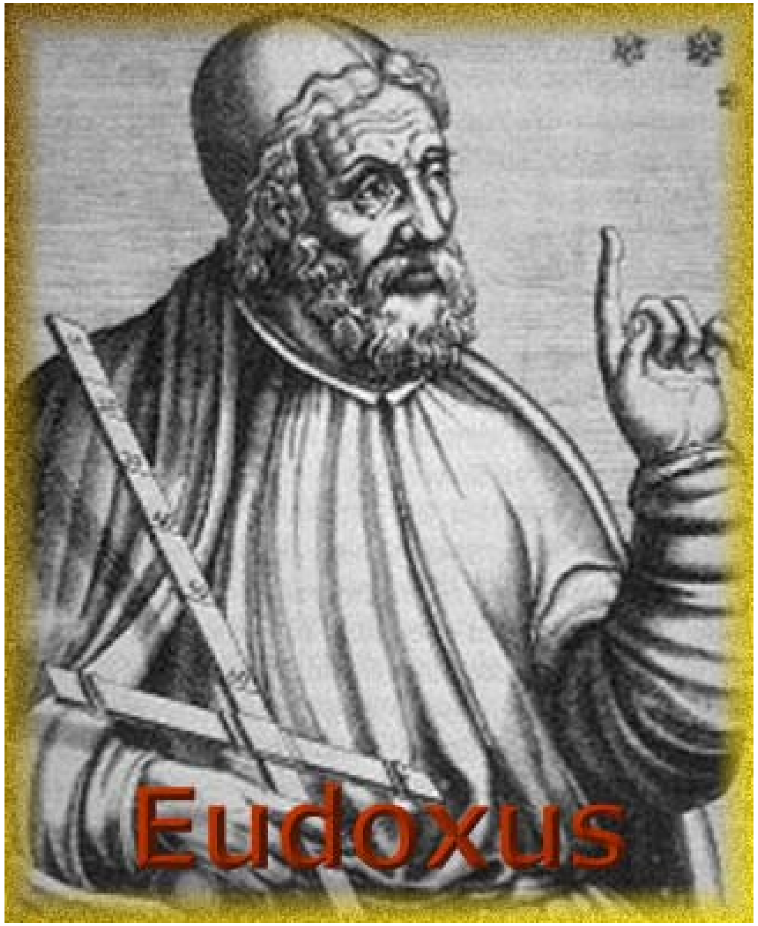
The theory of ratio by Eudoxos (the theory of quantity)

- A method which deals with the ratio of quantity (real number) by integers
- A : B = a : b means

with any given natural number m, n > 0

$$mA > nB \iff ma > nb$$

$$mA < nB \iff ma < nb$$



Eudoxos?(400 BC? – 350 BC?)

Dedekind cut (1858.11.24)

- A method which deals with real number that treats rational number as elements
- Real number x is the set of rational number

  $$\{r \mid r \text{ is rational and } r < x\}$$

- Hence if the real number equation $x = y$ holds,

  $$r < x \quad \Longleftrightarrow \quad r < y$$

  where r is any rational number.

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Dedekind(1831.10.6 – 1916.2.12)

$1 = 0.999999....$

because

$$r < 1 \quad \Longleftrightarrow \quad r < 0.999999....$$

where r any given rational number

Number System (19^{th} century)

- Natural number 0, 1, 2, 3, 4, ….
- Integer 0, ± 1, ± 2, ± 3, ± 4, ….
- Rational number $\frac{4}{3}$, $-\frac{7}{8}$, 0, ….
- Real number $\sqrt{2}$, e = 2.7182818…., π=3.141592653…., ….
- Complex number $\sqrt{-1}$, $\frac{-1 \pm \sqrt{-3}}{2}$, $\cos\frac{2\pi}{7} + \sqrt{-1}\sin\frac{2\pi}{7}$, ….

The discovery of a new number world (1897)

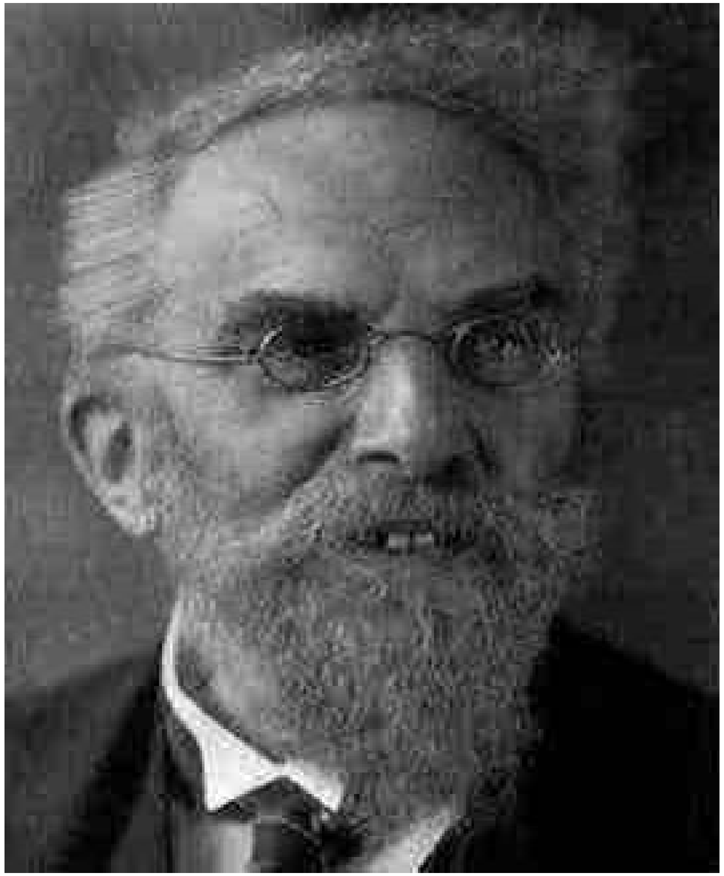
- Construction of real number based on the concept of rational number

  (Dedekind, Cantor)

- The discovery of a method to extend the number world
- using the same method,

the world of rational number was extended in a different direction

the great revolution in number theory (Hensel)



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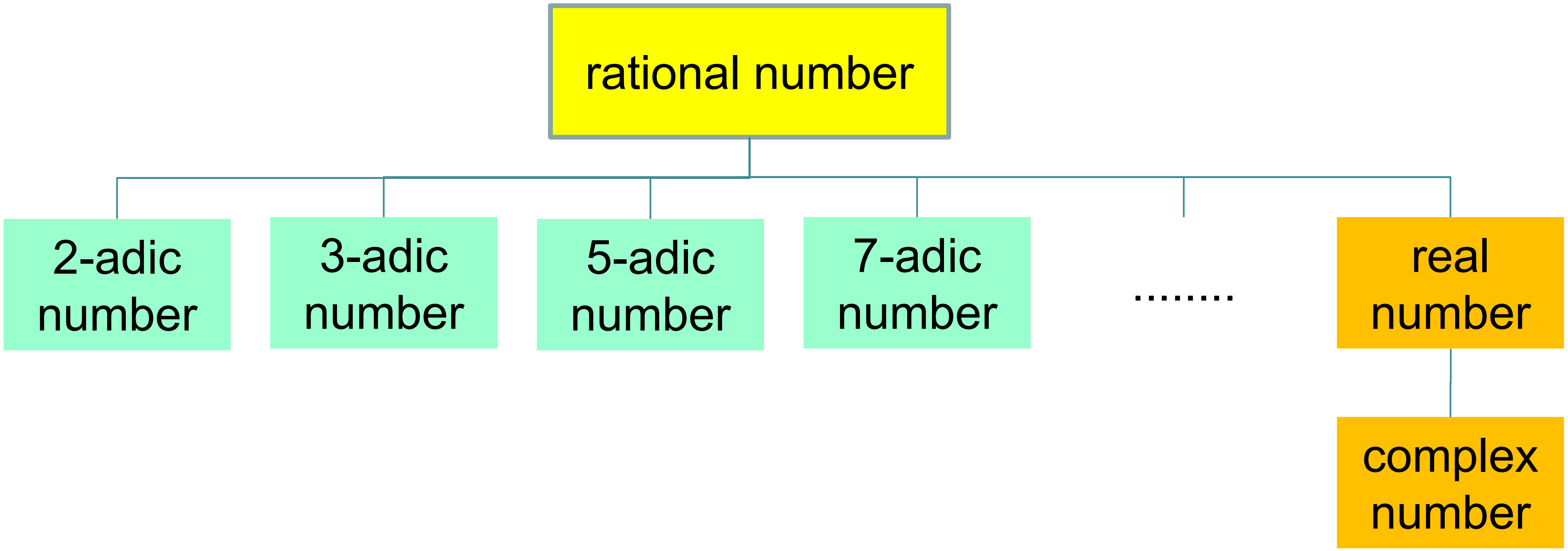
Hensel (1861.12.29–1941.6.1)

The number system(after 1897)



Method to extend the world of numbers

- real numbers are limits of rational numbers

  (Cantor 1872)

- corresponding to each prime number

  p = 2, 3, 5, 7, ...,There exists

one way to specify limit (topology),

and one can construct number system

(Hensel 1897)

there is no other way (Ostrowski)

The Way to specify limits .... topology

- Real number

  $1,\ 0.1,\ 0.01,\ 0.001,\ \ldots. \rightarrow 0$

- 7-adic number

  $1,\ 7,\ 49,\ 343,\ 2401,\ \ldots. \rightarrow 0$

7-adic number .... multiplied by 7 equals $\frac{1}{7}$

- (the set of all integers) multiplied by 7 equals

  (the set of all multiples of 7).

- (the set of all integers) is divided by 7 into equal parts

  (the set of all multiples of 7)

  (the set of all integers which leave the remainder of 1 when divided by 7),

  ....

  (the set of all integers which leave the remainder of 6 when divided by 7).

The advantages of extending the number world

- Equations become easier to solve

  $x^2 - a = 0$ (a integer)

- Concerned with rational number
- It cannot be determined until one decompose a into prime number

The advantages of extending the number world

$$x^2 - a = 0$$

- Concerned with real number

  if $a > 0$ solvable

  whereas $a < 0$ not solvable

The advantages of extending the number world

$$x^2 - a = 0$$

- Concerned with 7-adic number
- solvable if the remainder a leaves when divided by 7 is either of the following

$$1 = 1^2,\ 2 = 3^2 - 7,\ \text{or}\ 4 = 2^2$$

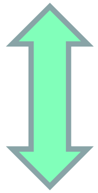
(Hensel)

not solvable if the remainder is 3, 5, or 6

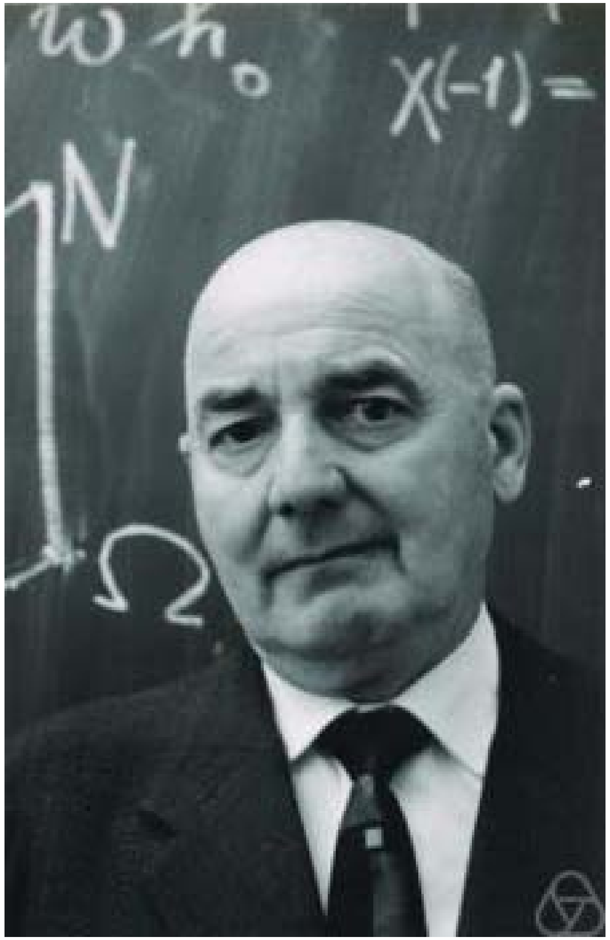
Local-global principle (Hasse principle 1921)

In the case of quadratic equation

- solvable as rational number



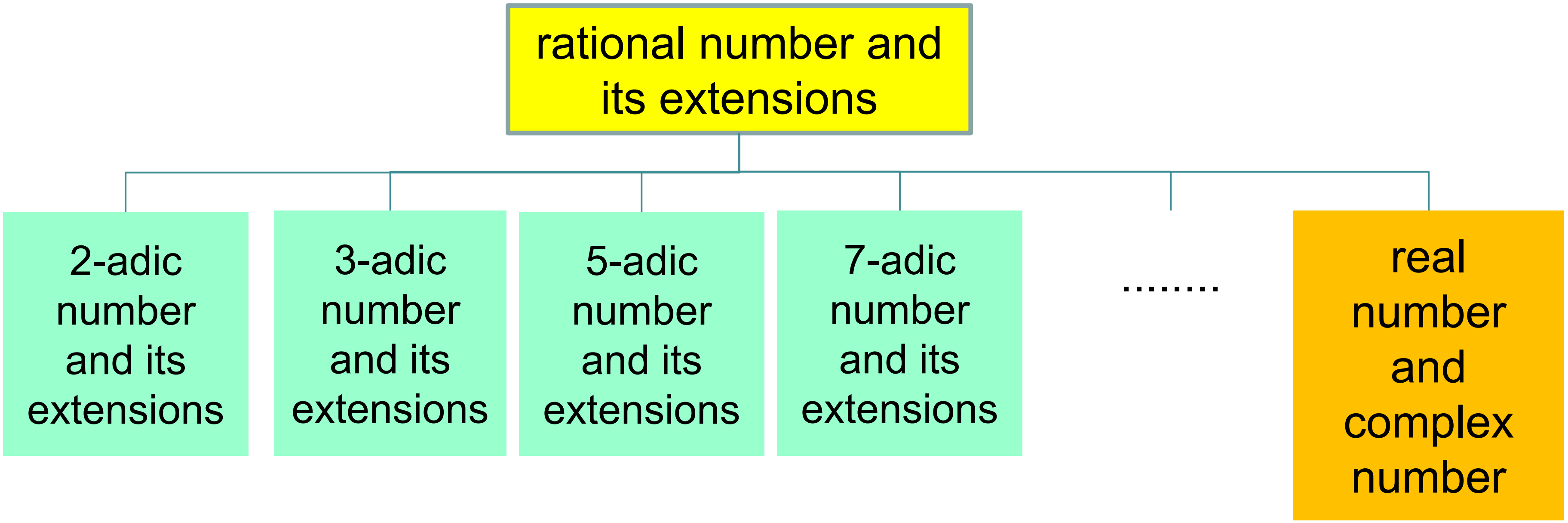
- solvable as p-adic number for all prime number p
- also solvable as real number



Hasse (1898.8.25–1979.12.26)

The number system

(the first half of the 20th century)



- The fundamental framework of modern number theory, that enables one to reach even for the settlement of Fermat’s last theorem, was originated.