

**Global Focus on Knowledge**  
"Creating Mathematics"  
-the challenge of mathematicians-

**2009 Winter Semester**

Lecture 13

**"Culture and Mathematics"**

Kazuo Okamoto

January 21



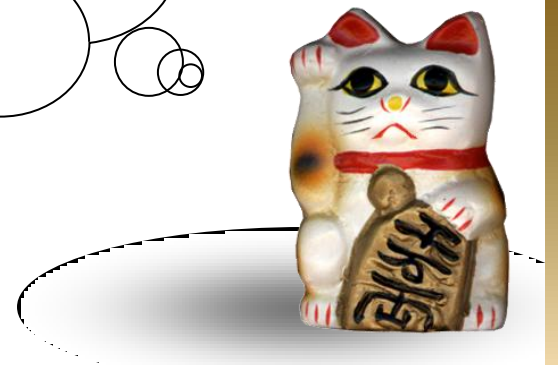
# Digression (Common sense) about mathematics

- ❁ Seven Liberal arts in Medieval Europe  
logic, grammar, rhetoric
- ❁ The Quadrivium  
geometry, arithmetic, astronomy, music
- ❁ Analysis and Synthesis





**Pierre de Fermat**  
**1601-1665**



Is  $F(n) = 2^{2^n} + 1$  a prime number?

$$F(0) = 2^1 + 1 = 3$$

$$F(1) = 2^2 + 1 = 5$$

$$F(2) = 2^4 + 1 = 17$$

$$F(3) = 2^8 + 1 = 257$$

$$F(4) = 2^{16} + 1 = 65537$$



$$F(5) = 2^{32} + 1 = 4294967297 \\ = 641 \times 670047$$



Demonstrated by  
Euler



$$F(5) = 2^{32} + 1$$

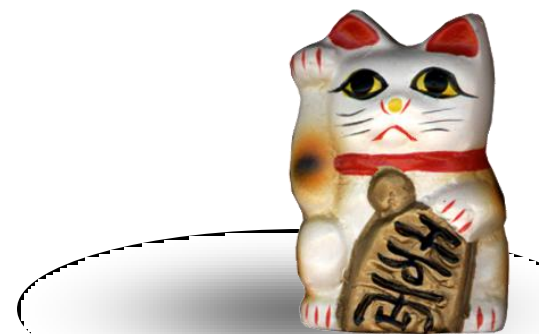
$$a = 2^7, b = 5, c = ab,$$

$$2^{32} = 2^{28}2^4,$$

$$2^{28} = a^4,$$

$$3 = 128 - 125 = a - b^3,$$

$$2^4 = 16 = 3 \times 5 + 1$$



$$F(5) = a^4 ((a - b^3)b + 1) + 1$$

$$= a^4 (c + 1) - c^4 + 1,$$

$$c^4 - 1 = (c - 1)(c + 1)(c^2 + 1)$$

F(5) can be divided by  $c+1=641$







# Leonhard Euler 1707-1783





# Proof for what?

$F(8)$  can be divided by 1238926361552897

$F(3310)$  can be divided by  $5 \cdot 2^{3313} + 1$

Caution! Though it is relatively easy to tell whether a given number is a prime or not, finding factors is much more difficult.

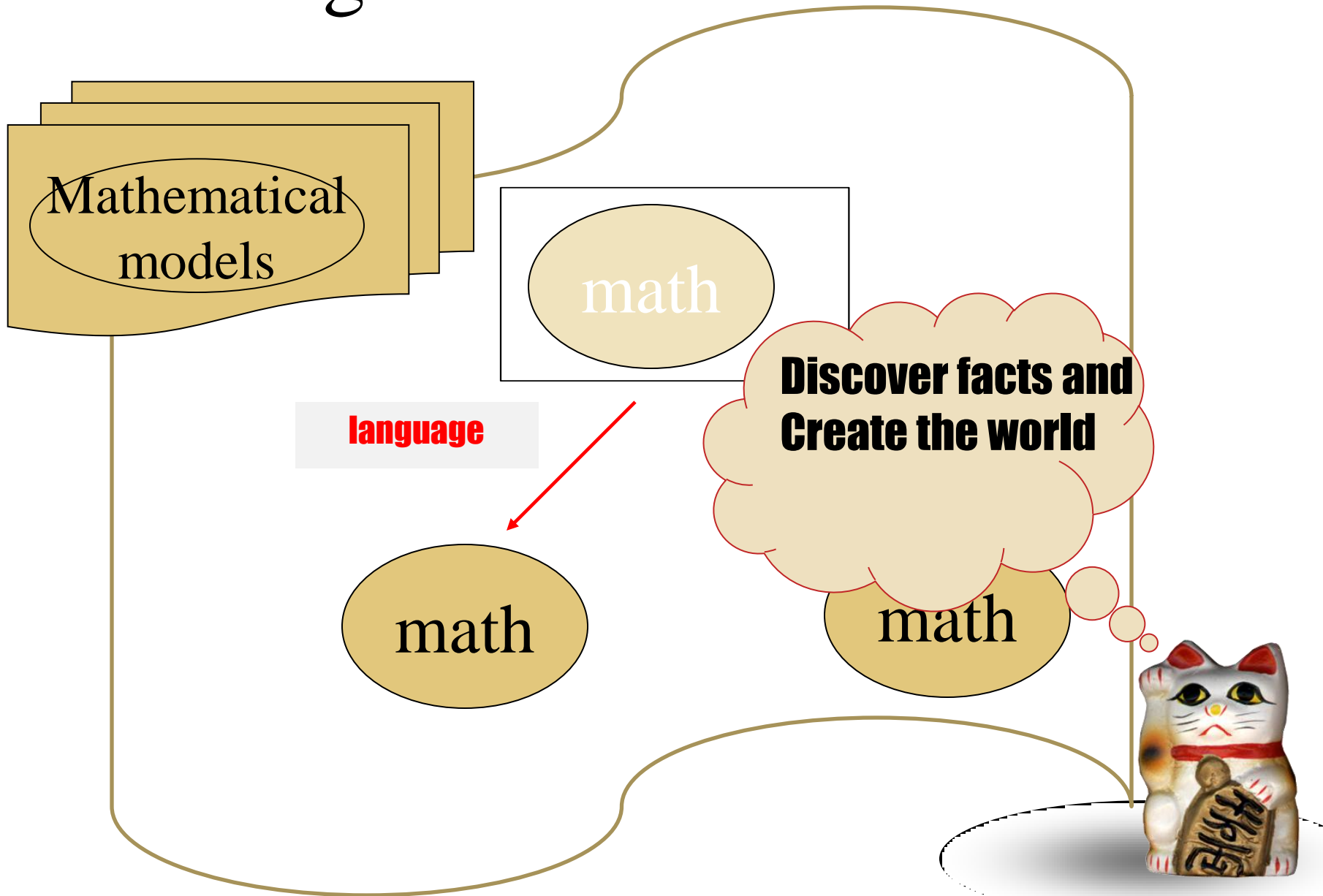


# What we expect for mathematics

- ❁ clear
- ❁ Inspiring
- ❁ Making us to have curiosity
  
- ❁ The joy of using math



# Creating mathematics



# Pure and applied mathematics

❁ Create math from math  
pure mathematics

❁ Create math from natural or social  
phenomena  
applied mathematics



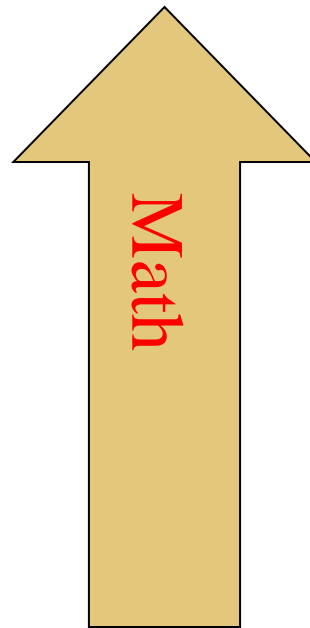
# Let me say one more thing about math

❁ From the students' viewpoint...

then

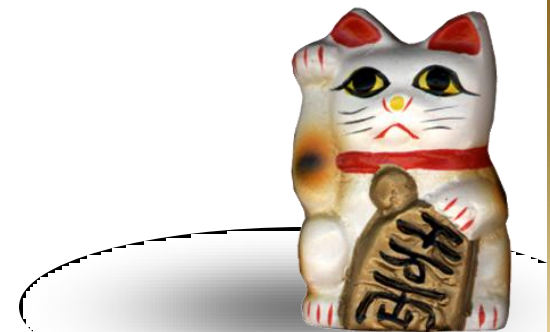
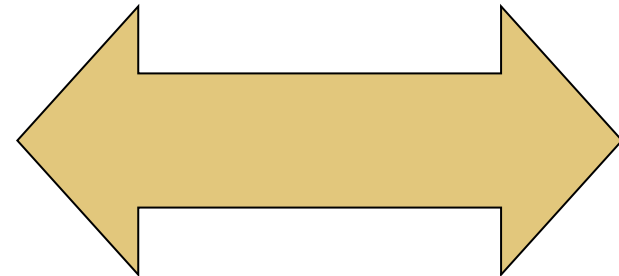
this

that



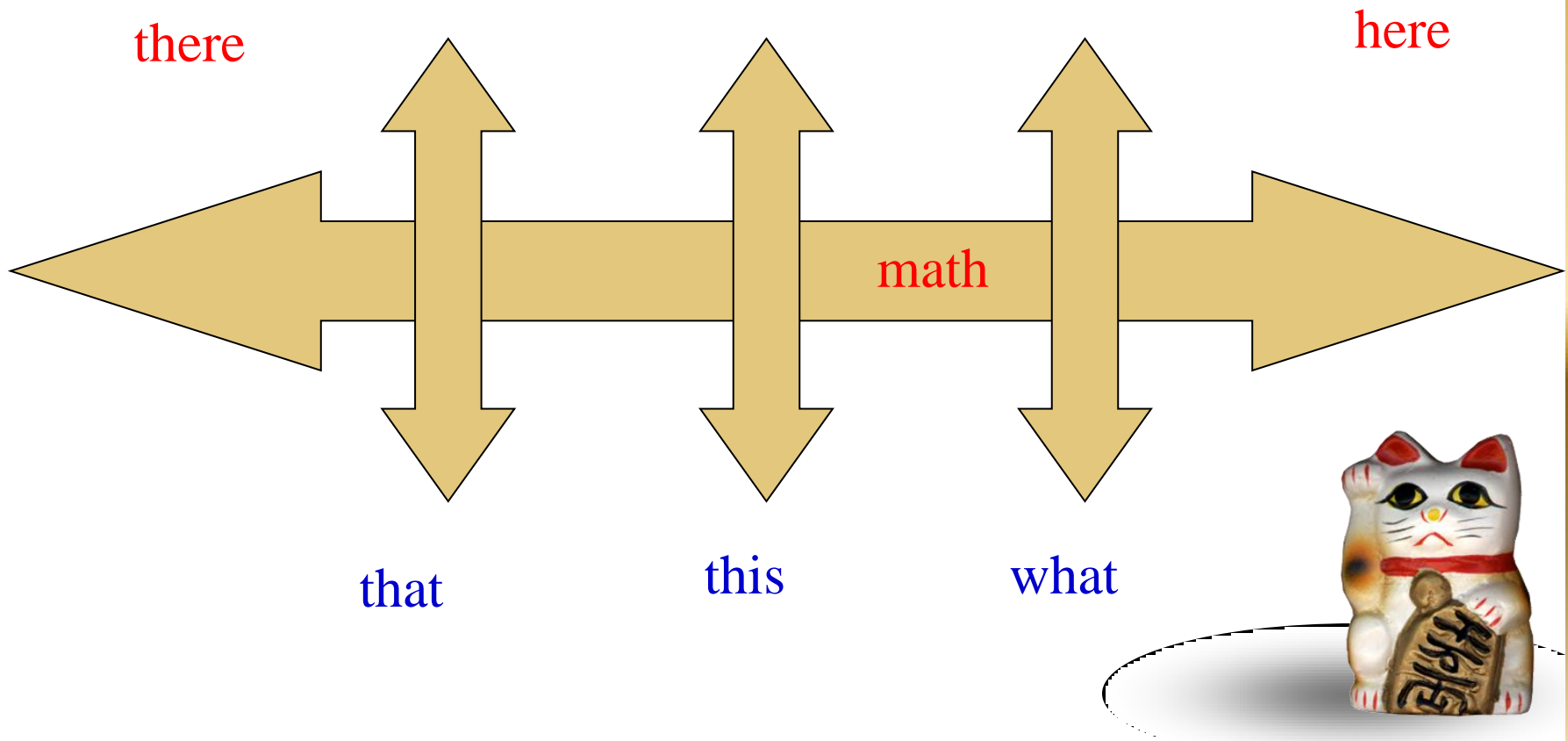
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# The role of math

❁ In science and technology...



# New way of use

Drawing curves  
Math as a fundamentals of  
CAD

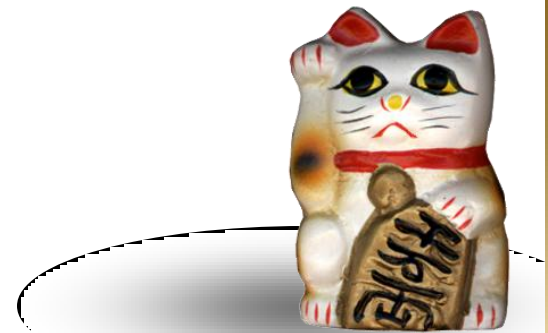
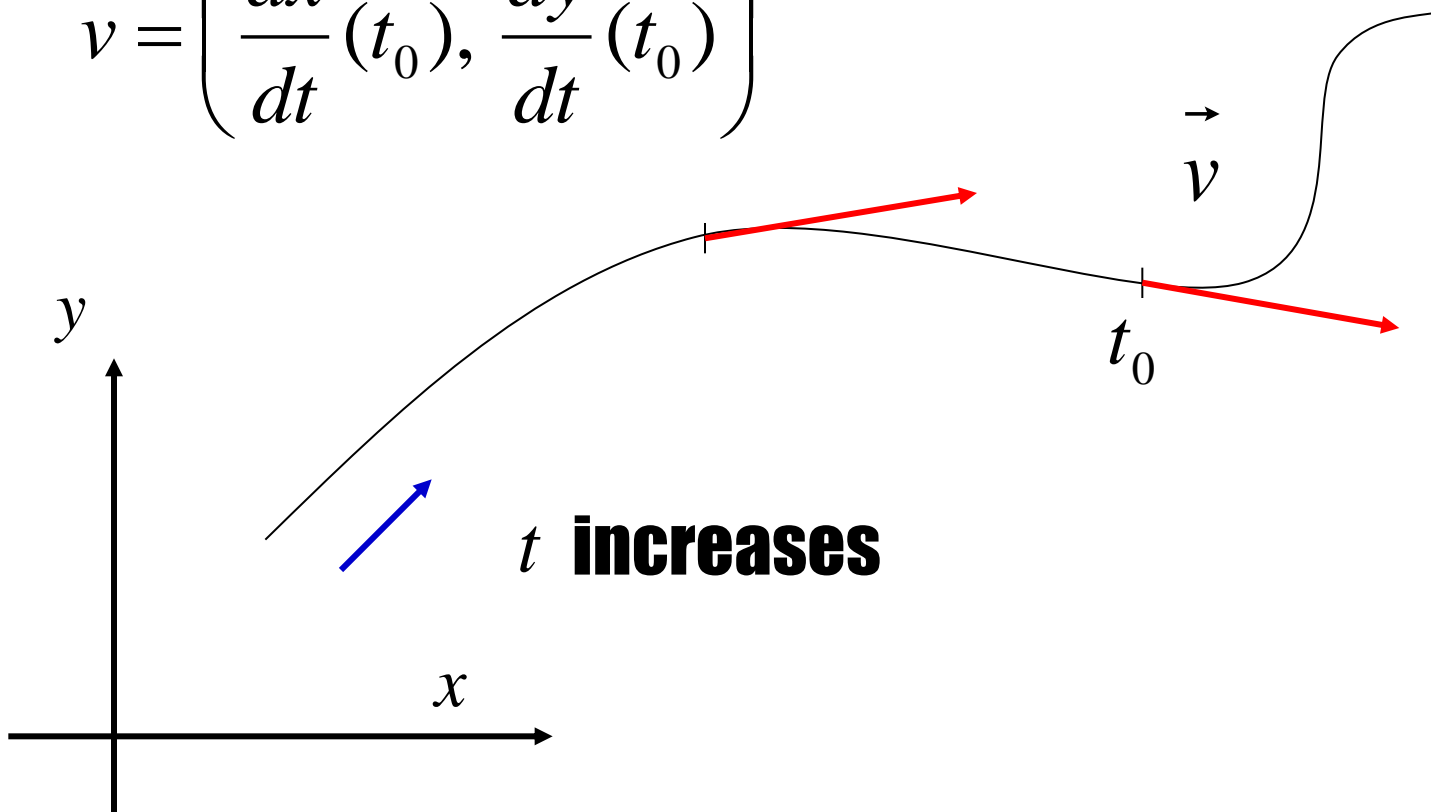
**Here's some math...**





# Tangent vector to $(x, y) = (x(t), y(t))$

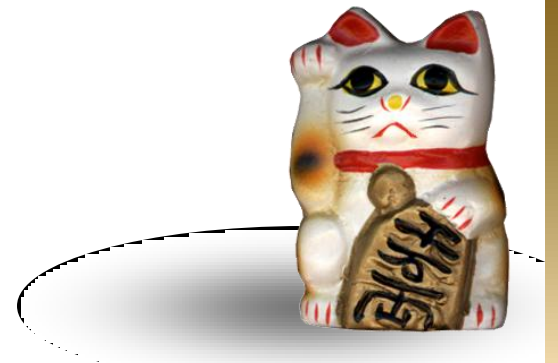
$$\vec{v} = \left( \frac{dx}{dt}(t_0), \frac{dy}{dt}(t_0) \right)$$



# 3rd order Bézier Curves

$$x = a_0(1-t)^3 + 3a_1t(1-t)^2 + 3a_2t^2(1-t) + a_3t^3$$

$$y = b_0(1-t)^3 + 3b_1t(1-t)^2 + 3b_2t^2(1-t) + b_3t^3$$



$$\frac{dx}{dt}(0) = -3a_0 + 3a_1$$

$$\frac{dx}{dt}(1) = -3a_2 + 3a_3$$

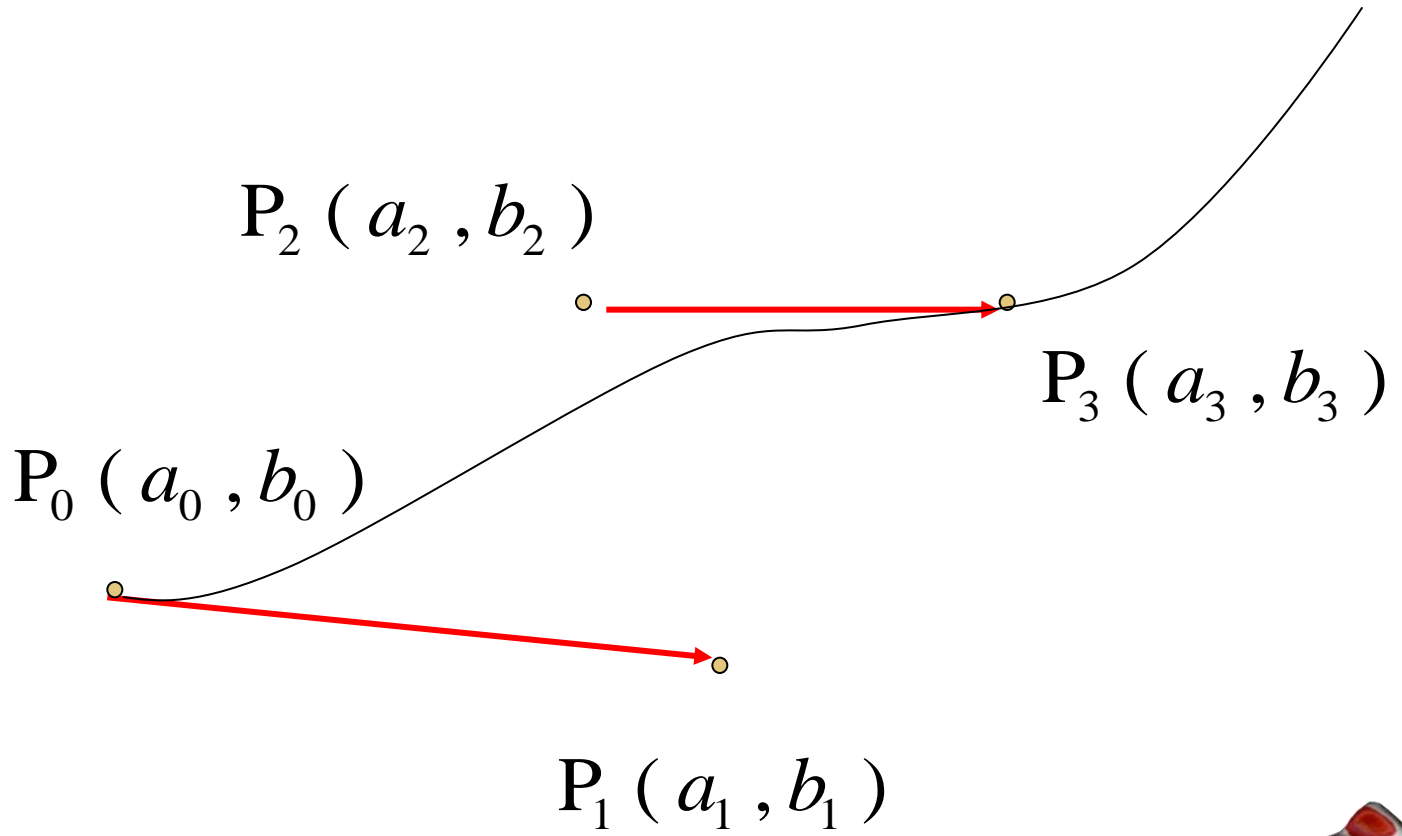
$$\frac{dy}{dt}(0) = -3b_0 + 3b_1$$

$$\frac{dy}{dt}(1) = -3b_2 + 3b_3$$

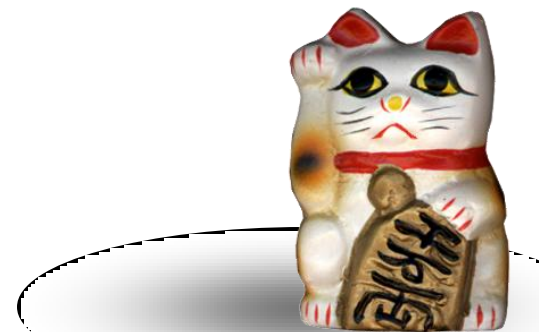
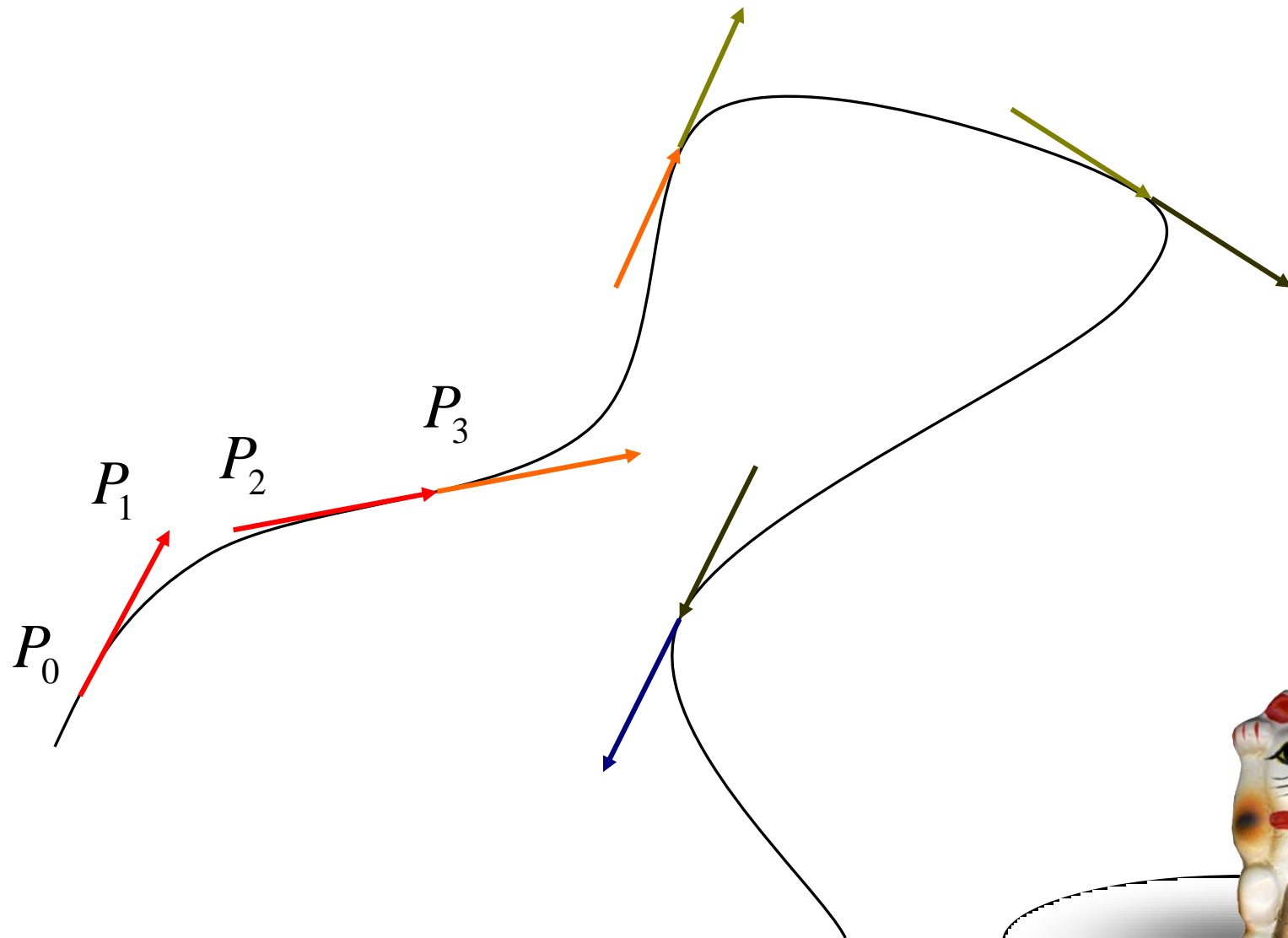
**Tangent vector at  $t = 0$  is**

$$-3 \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + 3 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$





# 3rd order Bézier Curves



$$x = a_0(1-t)^3 + 3a_1t(1-t)^2 + 3a_2t^2(1-t) + a_3t^3$$

$$y = b_0(1-t)^3 + 3b_1t(1-t)^2 + 3b_2t^2(1-t) + b_3t^3$$

$$u^3 + 3uv^2 + 3u^2v + v^3 = (u+v)^3$$

より

$$(1-t)^3 + 3t(1-t)^2 + 3t^2(1-t) + t^3 = (1-t+t)^3 = 1$$

factorization



$$x = a_0(1-t)^3 + 3a_1t(1-t)^2 + 3a_2t^2(1-t) + a_3t^3$$

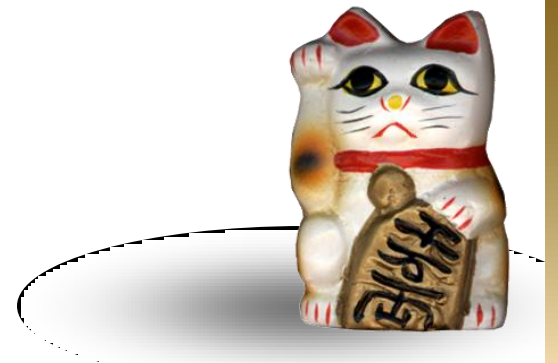
$$a = a(1-t)^3 + 3a t(1-t)^2 + 3a t^2(1-t) + a t^3$$

$$x - a =$$

$$(a_0 - a)(1-t)^3 + 3(a_1 - a)t(1-t)^2 + 3(a_2 - a)t^2(1-t) + (a_3 - a)t^3$$

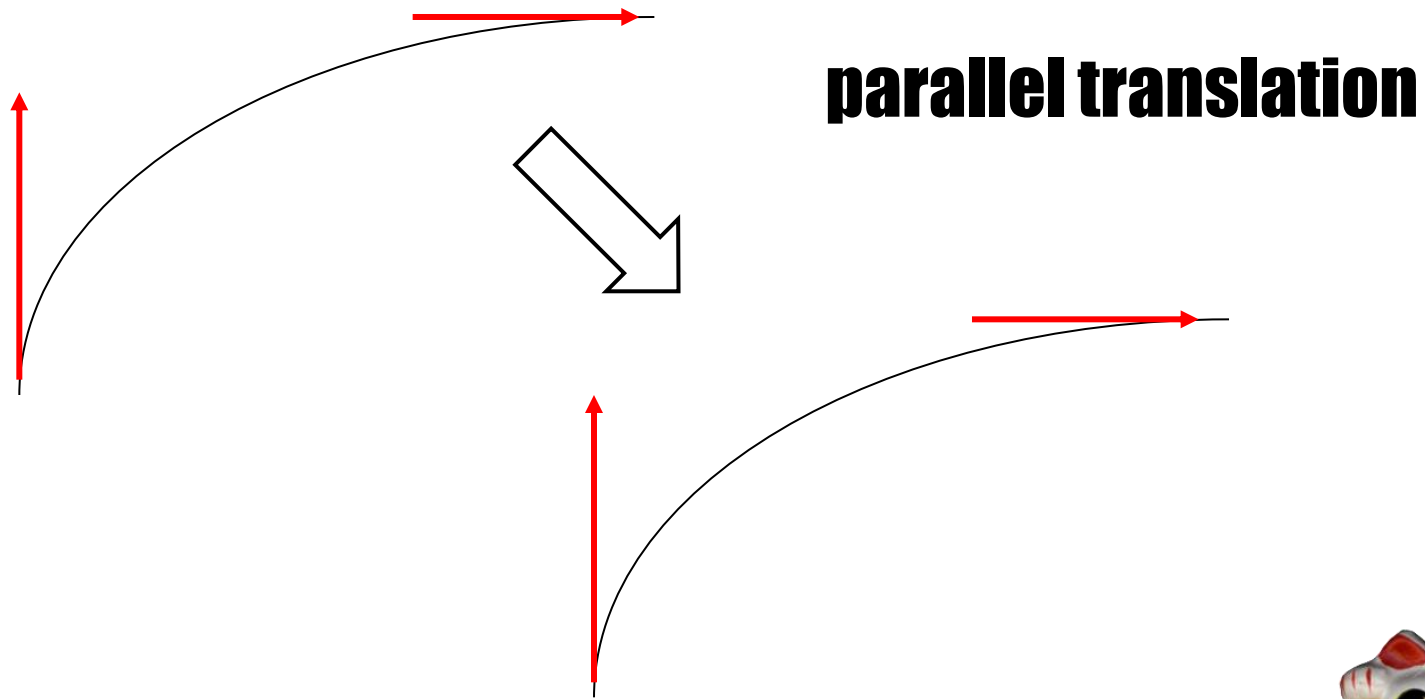
$$y - b =$$

$$(b_0 - b)(1-t)^3 + 3(b_1 - b)t(1-t)^2 + 3(b_2 - b)t^2(1-t) + (b_3 - b)t^3$$





**The figure does not vary when it is parallelly transported**





[http://de.wikipedia.org/w/index.php?title=Datei:Avignon\\_Panorama.jpg&filetimestamp=20090930184559](http://de.wikipedia.org/w/index.php?title=Datei:Avignon_Panorama.jpg&filetimestamp=20090930184559)

# Pierre Bézier

## 1910-1999

Photo from Systeme d'information sur le patronat français  
[http://sippaf.ishlyon.cnrs.fr/Database/Acteurs\\_fr.php?ID=AC000008168](http://sippaf.ishlyon.cnrs.fr/Database/Acteurs_fr.php?ID=AC000008168)



# Learn from the history first

- ❁ In principle, my policy in teaching is that much attention should be paid to the principles of nature and that all the human activities should be **BASED ON** two things, namely, number and reason.
- ❁ Generally speaking, to study something, it is necessary to get rid of the unnecessary part of the object and to get to the core of things. After that, number of topics to discuss will be lessened and you can get to the center of the problem.

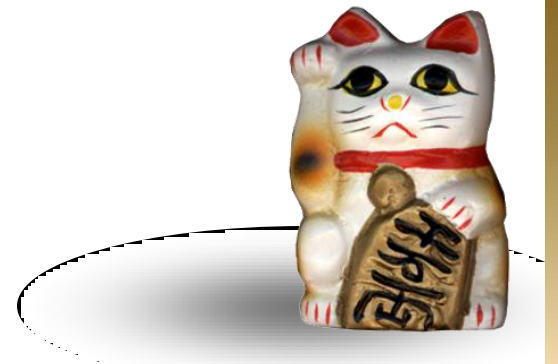


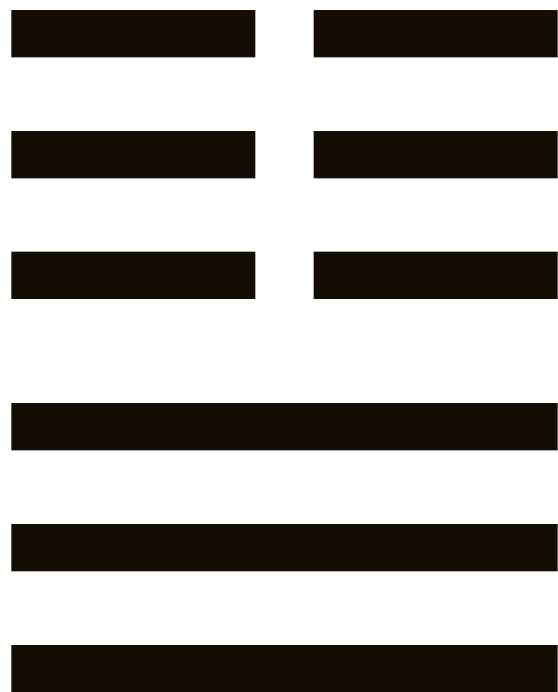
# What is Learning?



## Taegukgi

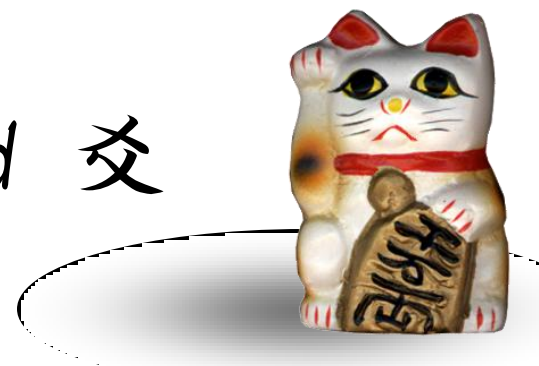
Taiji and Hexagram





# 秦 (Qin)

卦 (Hexagram) and 爻



周易上經乾傳第一  
 乾元亨利貞  
 初九潛龍勿用  
 九二見龍在田利  
 九三君子終日乾  
 九四或躍上則進  
 九五利見  
 上六亢龍勿用

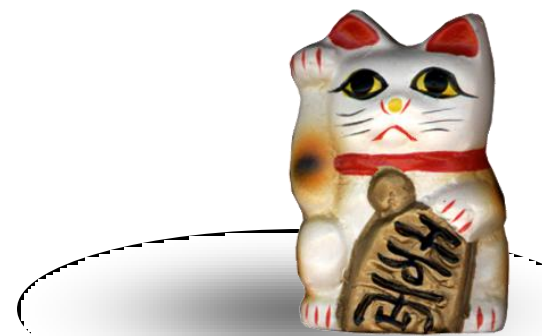
乾卦 天  
 乾元 利貞  
 初九 潛龍勿用  
 九二 見龍在田利  
 九三 君子終日乾  
 九四 或躍上則進  
 九五 利見  
 上六 亢龍勿用

王弼註  
 乾者天之象也  
 利貞者天之德也  
 初九潛龍勿用  
 九二見龍在田利  
 九三君子終日乾  
 九四或躍上則進  
 九五利見  
 上六亢龍勿用

# I Ching



# 學 教



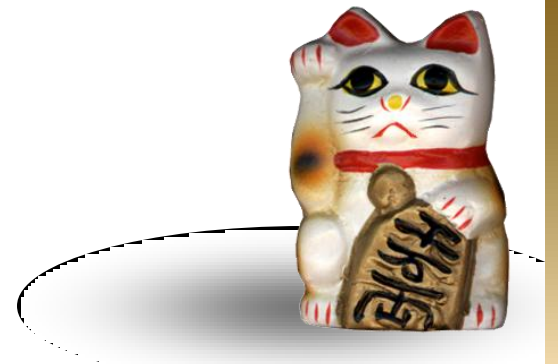


I cannot teach you  
more than I have,  
but you can learn  
more than  
I can teach you.



# Scientific evolution

Change the notation  
Change the view



# Equation of harmonic oscillation

$$\frac{d^2 x}{dt^2} + \omega^2 x = 0$$

one variable, two-order  
Differential equation

$$v = \frac{dx}{dt}$$

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\omega^2 x$$

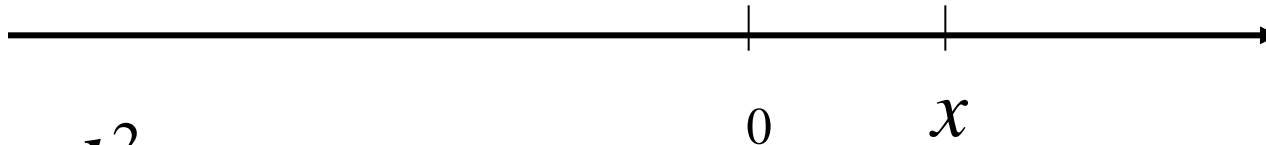
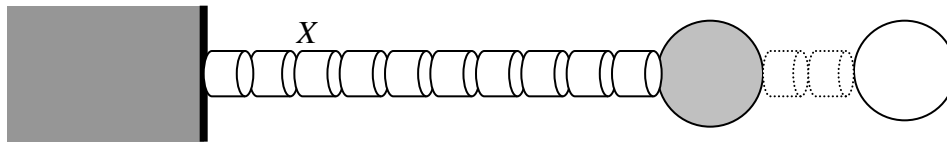
**Simultaneous  
differential  
equation**



# Hooke's law

$$\frac{d^2 x}{dt^2} + \omega^2 x = 0$$

$$-kx$$



$$m \frac{d^2 x}{dt^2} = -kx$$

$$\omega^2 = \frac{k}{m}$$



# Total energy

$$H = \frac{1}{2} v^2 + \frac{1}{2} \omega^2 x^2$$

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\omega^2 x$$

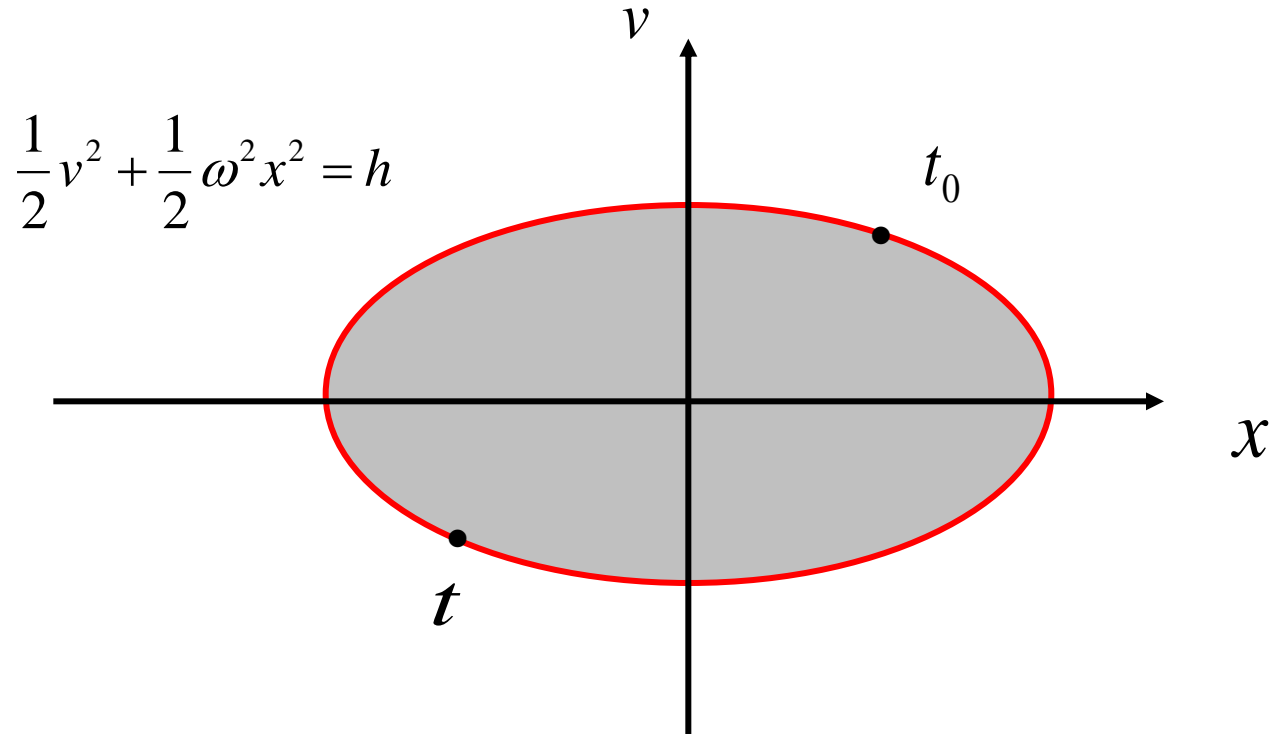
$$\frac{dH}{dt} = v \frac{dv}{dt} + \omega^2 x \frac{dx}{dt} = v(-\omega^2 x) + \omega^2 x v = 0$$

$$h = \frac{1}{2} v(t_0)^2 + \frac{1}{2} \omega^2 x(t_0)^2$$

$$H = h$$



$H = h$  Indicates an ellipse in xv-plane



Oscillating in the x-direction

It is called  
phase  
space



# Hamiltonian

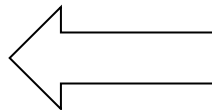
$$H = \frac{1}{2} v^2 + \frac{1}{2} \omega^2 x^2$$

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\omega^2 x$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial v}$$

$$\frac{dv}{dt} = -\frac{\partial H}{\partial x}$$



Fundamental form of  
The equation of  
dynamics



# Harmonic oscillator

change the point of view

$$H = \frac{1}{2} p^2 + \frac{1}{2} x^2$$

Imagine that it is simply a sign and that for function  $\psi$ , only the form  $H\psi$  has meanings.

$$H\psi = -\frac{1}{2} \frac{d^2\psi}{dx^2} + \frac{1}{2} x^2\psi$$





$$H = \frac{1}{2} p^2 + \frac{1}{2} x^2 \quad \text{for } H\psi$$

$$p^2\psi \Rightarrow -\frac{d^2\psi}{dx^2} \quad x^2\psi \Rightarrow x^2\psi$$

$x^2$  would simply equal to x squared, but

$p^2$  Would be a differential operator

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2$$

$$p \Rightarrow \frac{1}{i} \frac{d}{dx}$$

It is called  
quantization



# Schroedinger equation

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2$$

$$H\psi = E\psi$$

The motion of springs  
leads to  
(though formally)  
Quantum mechanics

$$\int_{-\infty}^{\infty} |\psi|^2 dx < \infty$$

The value  $E$  satisfying this condition is called Eigenvalue.



$$\psi_0 = \exp\left(-\frac{1}{2}x^2\right)$$

$$H\psi_0 = \frac{1}{2}\psi_0$$

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2$$

$$H\psi = E\psi \quad \int_{-\infty}^{\infty} |\psi|^2 dx < \infty$$

This is true if and only if  $E_n = \frac{1}{2} + n$

$\psi_n = H_n(x)\psi_0$        $H_n(x)$  Is a polynomial of  $x$



# Young people's ability needed

- ❁ University's needs

- ❁ Company's needs

- ❁ Specialty

- ❁ general ability

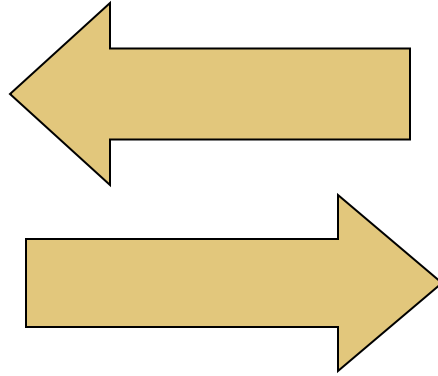
ability to communicate  
with others



# presentation



sender



receiver

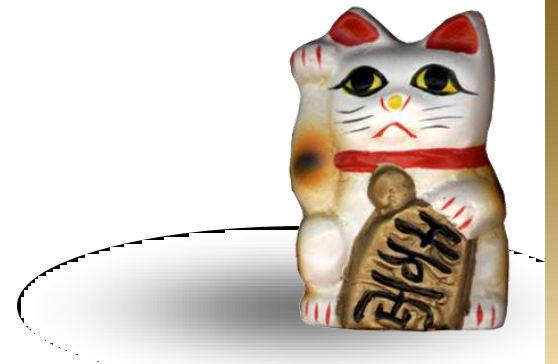
$$P = C \times T \times M$$

$C$	contents	
$T$	technique to present	$0 \leq T \leq 1$
$M$	receiver's motivation	$M \leq 1$



# Then how about me?

- ❁ I find problems from math, use math, and express mathematically
- ❁ an ordinary pure mathematician (I think)



# My math

Integrable systems  
Painlevé equations



# A little talk about my major

Theory of integrable systems

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x}$$

This is the KdV equation that represents  
Nonlinear waves.

It is related to Tsunami.





In the latter half of 20<sup>th</sup> century,  
A new viewpoint in mathematics is revived.

Keywords

Chaos, Soliton, Fractal

KdV equation is a Soliton equation.

To grasp the concept,  
Let us express KdV equation in  
A digital form.



000011110000110000100000000000000000000000

There are not any number other than 0 on left or right side.

rules

From the left side, 1 is moved to the point of next 0.  
We write 0 in the blank.



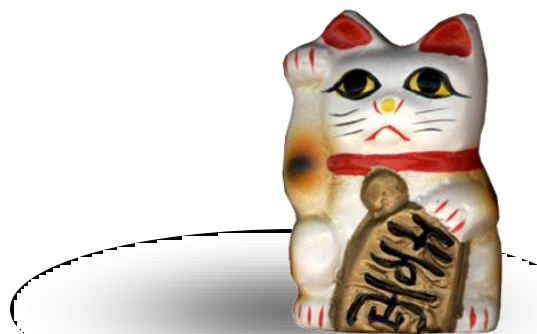
0000111100001100001000000000000000000000000  
00000000111100110001000000000000000000000000

Soliton in ball in box



000011110000110000100000000000000000000000  
000000001111001100010000000000000000000000  
000000000000110011101100000000000000000000

4 block then take over...

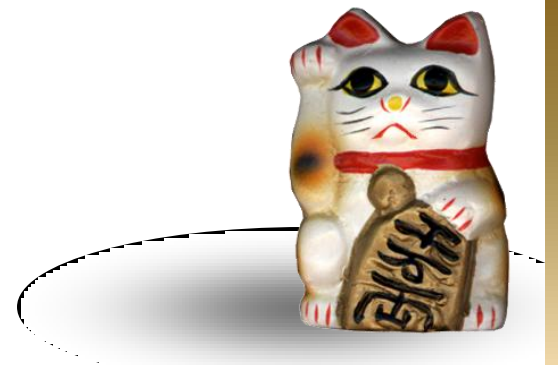


000011110000110000100000000000000000000000  
000000001111001100010000000000000000000000  
000000000000011001110110000000000000000000  
000000000000000110001001111000000000000000

Then get ahead



0000111100001100001000000000000000000000  
0000000011110011000100000000000000000000  
0000000000001100111011000000000000000000  
0000000000000011000100111100000000000000  
0000000000000000110010000011110000000000

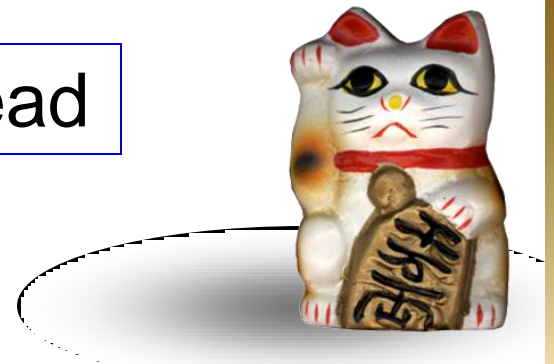


0000111100001100001000000000000000000000  
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000000000000000000110010000011110000000000  
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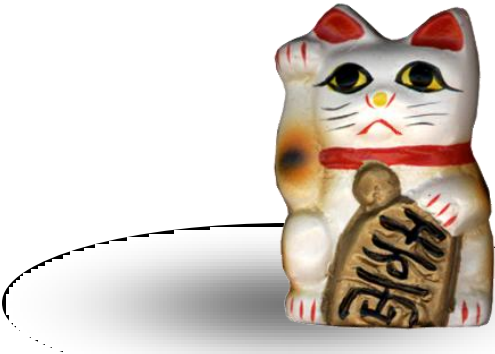
2 blocks take over and get ahead





```
000011110000110000100000000000000000000000000000
000000001111001100010000000000000000000000000000
000000000000110011101100000000000000000000000000
000000000000001100010011110000000000000000000000
000000000000000000110010000011110000000000000000
000000000000000000001101000000001111000000000000
000000000000000000000010110000000000111100000000
000000000000000000000000100110000000000000111100
00000000000000000000000000100110000000000000001
```

the circumstance is reversed





$$\frac{d^2 y}{dx^2} = 2y^3 + xy + \alpha$$

This is the Painlevé equation.





**Paul Painlevé**  
**1863-1933**

The first mathematician to take a plane  
Election for Minister and President defeated



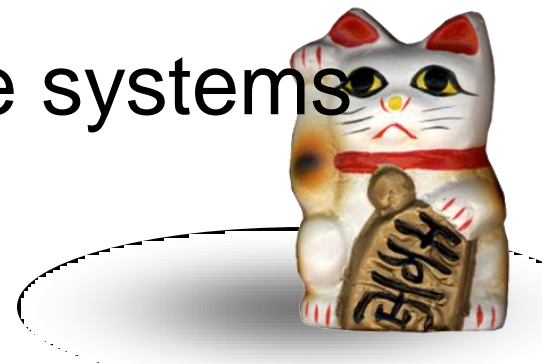
# Painlevé equation

- ❁ Find a “new” transcendental function defined by differential equations
  - containing hypergeometric functions and elliptic functions
  - the object “nonlinear integrable systems”
- ❁ Specialization rather than generalization
  - dual to each other
- ❁ But...



# History of Painlevé equation

- ✿ Around 1900  
discovery of Painlevé equation
- ✿ 1907 isomonodromic deformation
- ✿ 1912年 Garnier systems
- ✿ 1976年頃 two dimensional Ising model  
around here Soliton, Chaos, ...  
revival of nonlinear integrable systems



# Present state of Painlevé equation

- ✿ Recognized once as special function
- ✿ Lead by mathematics in physics. Where is math?
  - geometry in initial value space
  - transformation groups and root system
  - structure of special solution
- ✿ Garnier systems and extension to Noumi-Yamada systems



# “pure” math and “applied” math

- ❁ Both Pure math and Applied math are part of mathematics if mathematical science includes interfaces through mathematics.
- ❁ Though it is unnecessary for all mathematician to look out of math, but isolation is no good.
- ❁ A world around math is important, such as application and education.



The most important  
thing

Do you like math?

