Global Focus on Knowledge

“Creating mathematics” lecture 12

How to Distinguish Different Shapes, and the Mathematical Point of View

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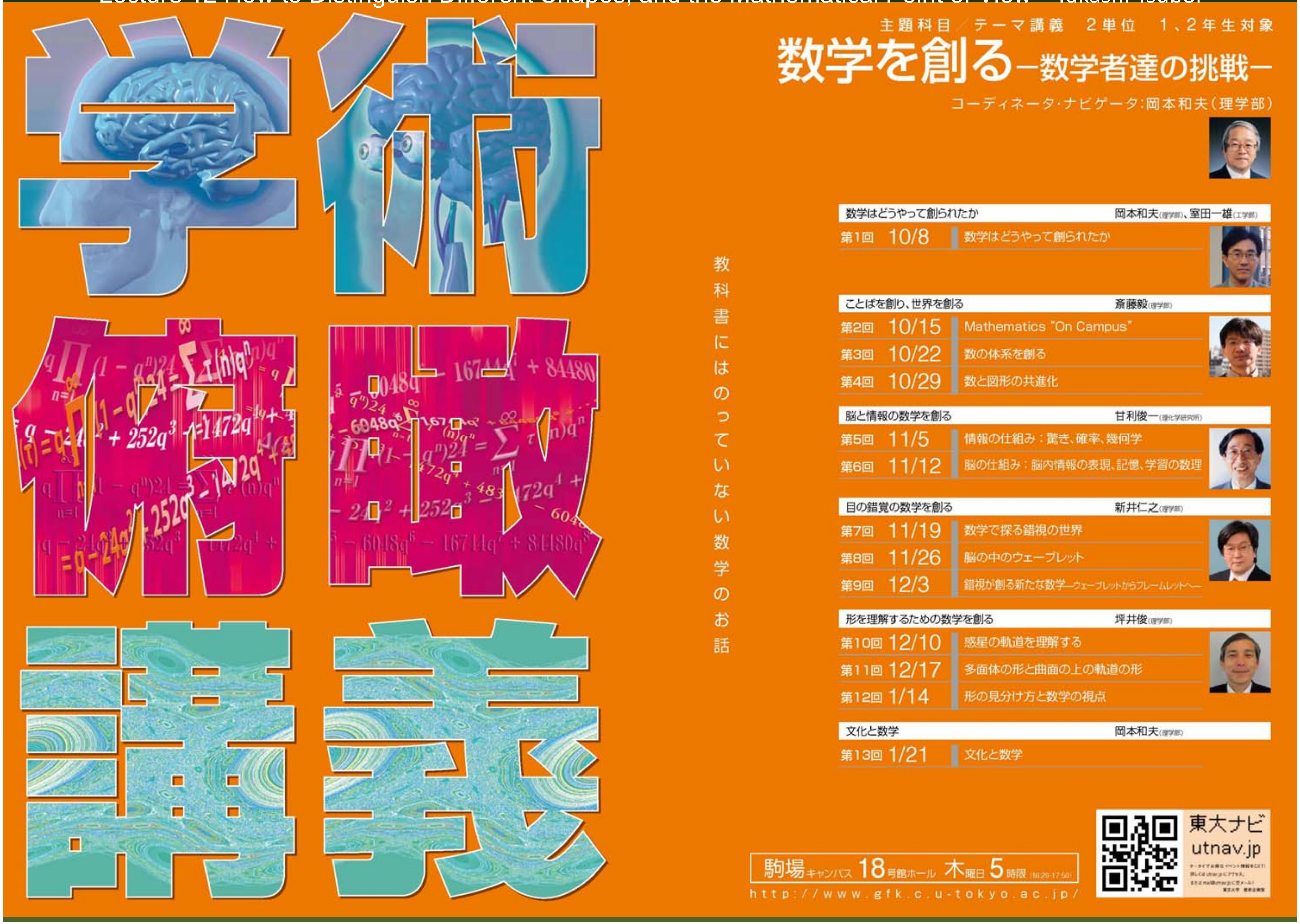
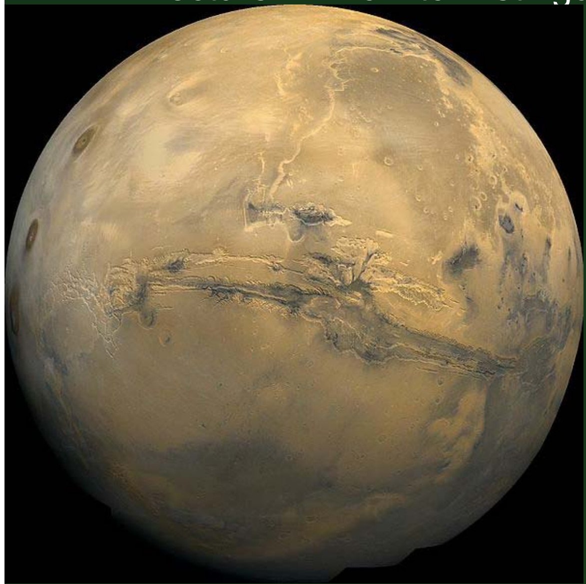
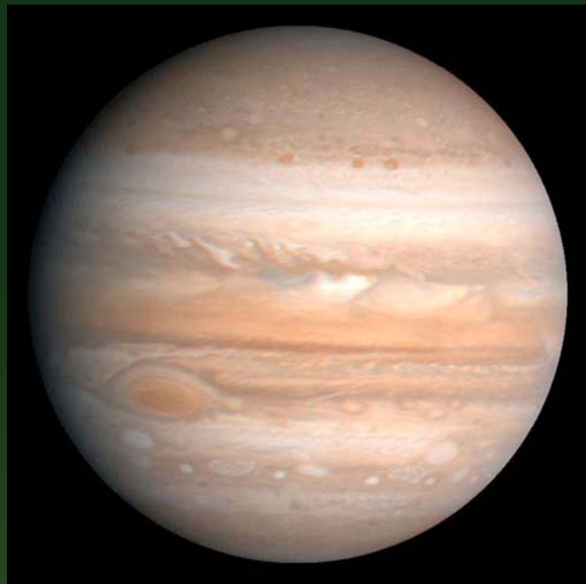
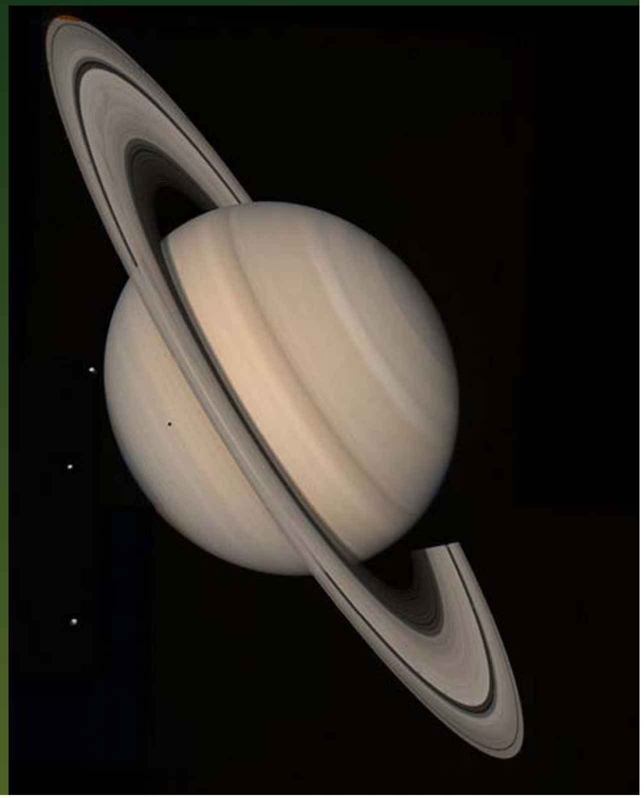
December 14th, 2010 Global Focus on Knowledge Creating Mathematics
Lecture 12 How to Distinguish Different Shapes, and the Mathematical Point of View Takashi Tsuboi



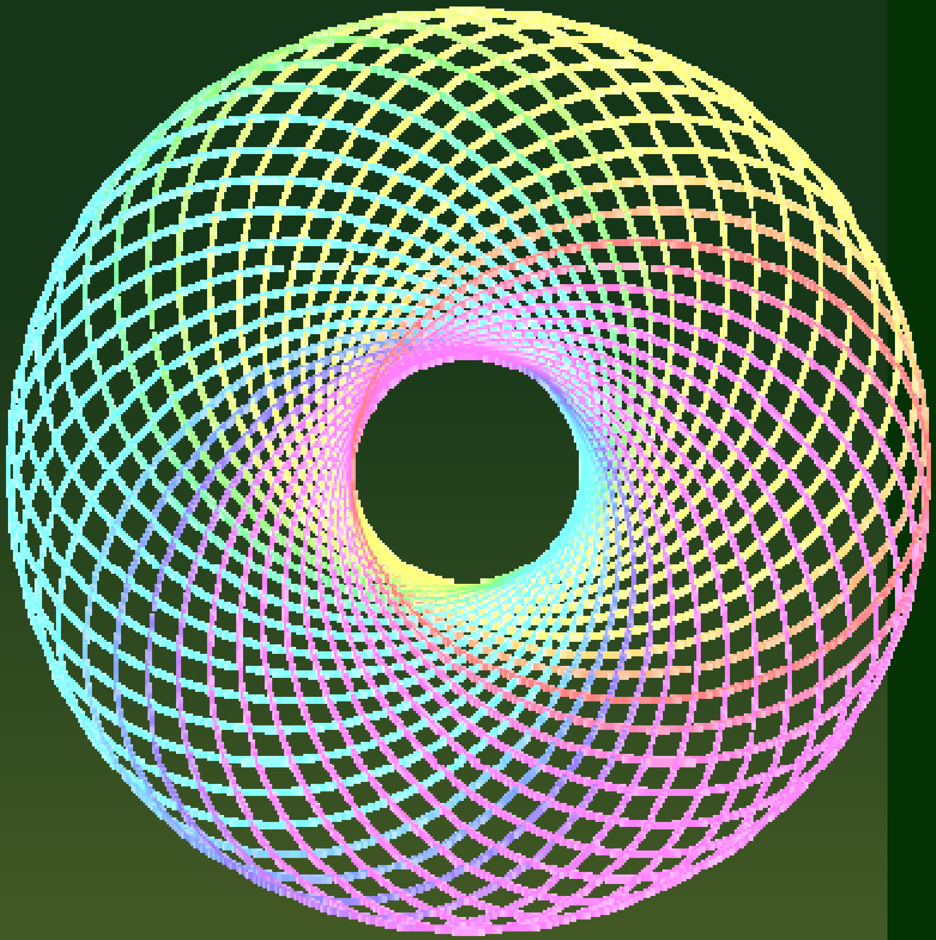


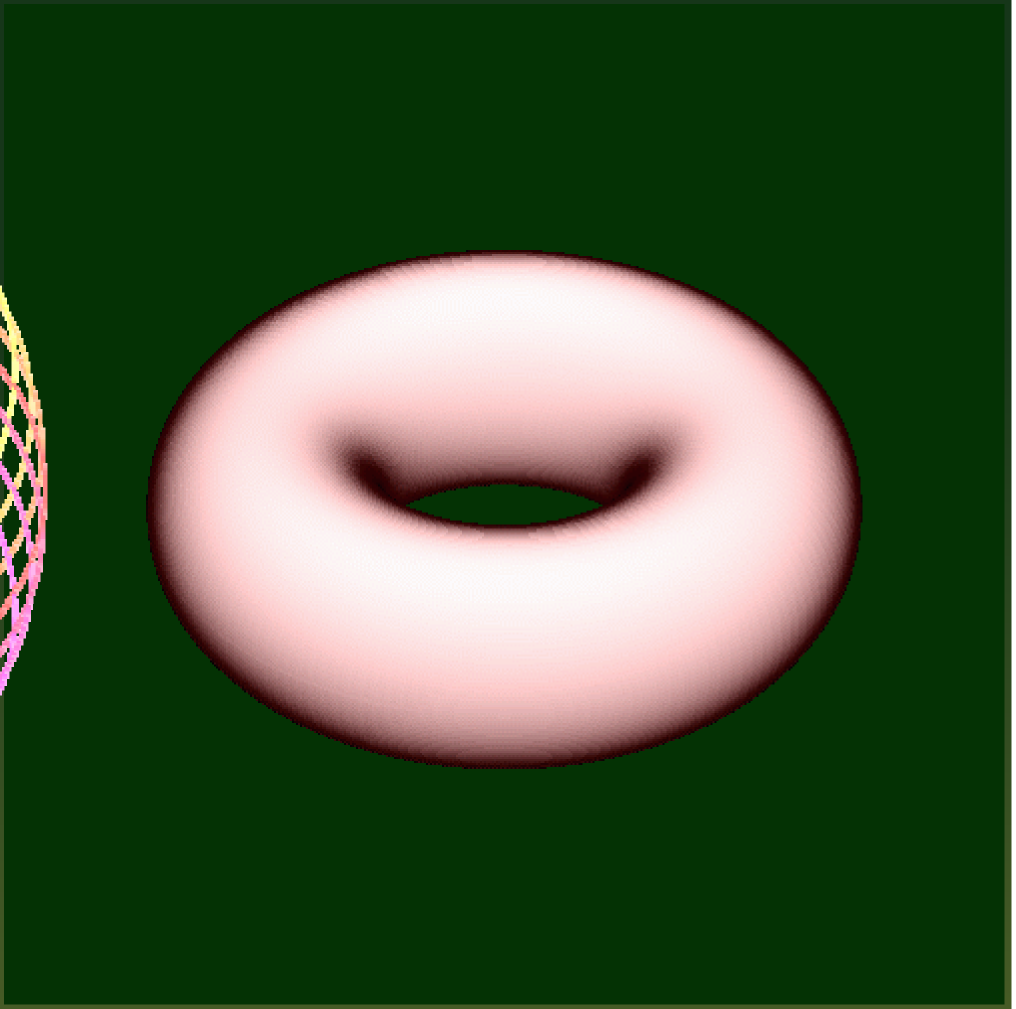




Planets

Reprinted from NASA Webpage





●Impressions

●I am impressed to see that even for simple polyhedrons we see in everyday life, the beautiful Euler polyhedron theorem always holds.

！！I want you to be impressed！！

●It was a surprise for me that Euler's polyhedron theorem is related to the orbits of planets.

●That a theorem of polyhedrons is applicable to curved surfaces is amazing.

！！That is why mathematics is interesting！！

●I couldn't grasp the concept of a torus.

！！Umm！！

●Torus is found in video games (RPG map, DQ, etc)

！！since mathematics is useful when creating video games, many game developers are familiar with such mathematical knowledge.

●Questions

●I have heard that figures like tori are not natural in Euclidean spaces, and I'm wondering what the true figure is like.

！！In fact, surface of a torus is one of the natural forms.

Another natural form is a figure that is flat in two directions, and on such figures, you would always return to the starting point.

●It is not possible to fill the spaces with Archimedean polyhedrons, whereas it is possible for diamond-shaped polyhedrons. How can we demonstrate this?

！！You can show the former paying attention to the angle between two faces, and as to the latter, play with toy！！

●I have heard the reason that planetary orbit do not change with time is due to another conserved quantity. Is it true?

！！It is due to the inverse square law！！

●Questions

●Do concave polyhedrons have any relations to planets or natural phenomena?

!! Surfaces of genus more than 2 are, when they are in their most natural form, concave at all points(though it cannot be realized in 3-dimensional spaces (Hilbert's theorem)). These can also be space of states. !!

●Are there any applications of curved surface study?

!! There are few mathematical researches that are not related to curved surfaces. In addition,… !!

●Are there any applications of polyhedrons?

!! All the figures we observe are curved surfaces or polyhedrons.

Designing automobiles, trains, and planes, and plastic products here and there are made using metallic molds, and it is necessary to understand the properties of minimal surfaces or the strength of curved resin.

Polyhedrons are also applied to studies of chemical reactions of interfaces of liquids and reactions of macromolecules. !!

●Questions

●Are there some more Platonic polyhedrons other than the five ones?

！！We can demonstrate that with Euler's polyhedron theorem！！

●Euler characteristic of the Mobius' ring is zero. Is the property of Mobius' ring similar to that of a torus?

● ！！That's true. On the surface of Mobius' ring, you can draw a flow that has no stationary points, and which is always perpendicular to the boundary.！！

What are the problems that Poincaré（1854－1912） was engaged in?

- In Poincaré's era, there was a breakthrough in understanding surfaces which are two-dimensional manifolds (or one-dimensional complex manifolds).

- Riemann's mapping theorem became the present form.

- "Riemann's mapping theorem"
  a one-dimensional, simply-connected complex manifold is, complex analytically, the same as one of three objects, namely, a Riemann surface, a complex plane, or a unit open disk.

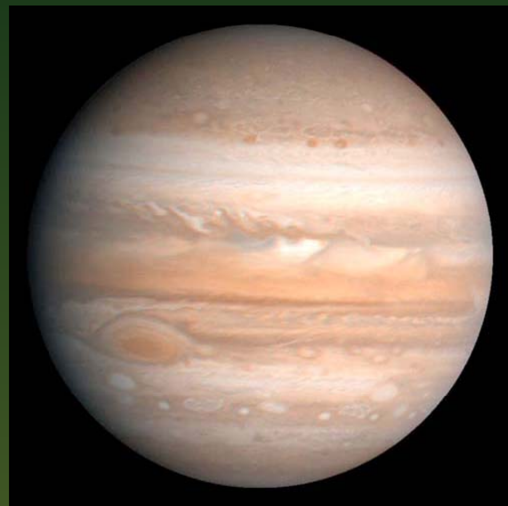
- Little was yet studied about higher dimensional manifolds.
- It was around the middle of the 20th century that the debates on the definition of manifolds were settled.

Poincaré did his mathematical researches in such an era:

- Three-body problem is difficult.
- Even for three-body problem on flat surface, the space of coordinates and velocity of three points are a $(2+2)\times 3=12$ dimensional space.
- If we take the coordinate which has its origin at the center of mass of three points, the dimension of space of state becomes 8.
- Due to conservation of angular momentum and energy, dimension of the space of state become 6, but this is too high a dimension to understand.

- For some time, we will study circular restricted three-body problem.
- Assume that the sun and Jupiter is moving circularly around their center of mass,
and we try to describe the motion of the third planet.

National Astronomical Observatory of Japan



BASIC codes for the three-body restricted problem

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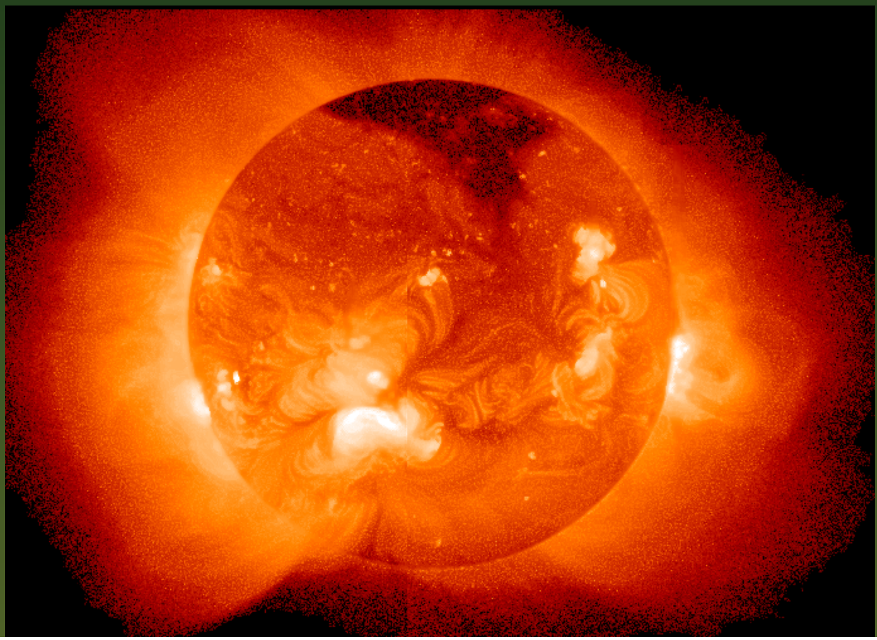
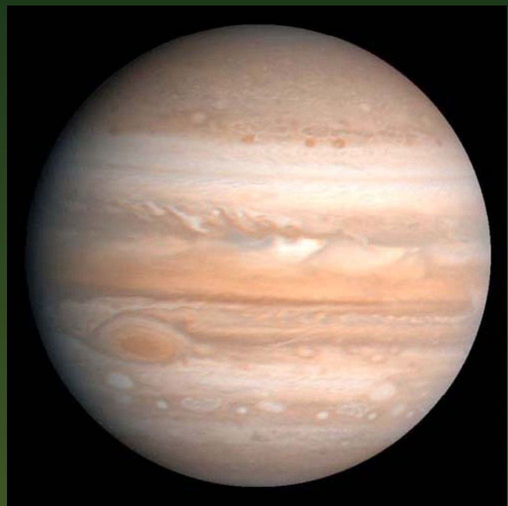




Basic codes for the circular restricted three-body problem

- **In circular restricted three-body problem, we can take the coordinate system in which circularly moving Sun and Jupiter are fixed.**
- **In this coordinate system, there are centrifugal and Coriolis force other than gravitational force.**

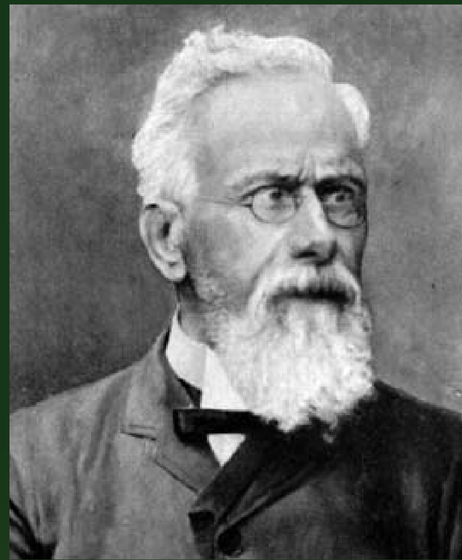
Basic codes for restricted three-body problems in rotating frame.

- **the space of state that represents the third planet is 2+2=4 dimensional space.**
- **This motion has a constant quantity called Jacobi Integral that corresponds to energy, the dimension of the space of state is 3.**
- **By studying flows on three-dimensional manifolds, we can understand circular restricted three-body problem.**

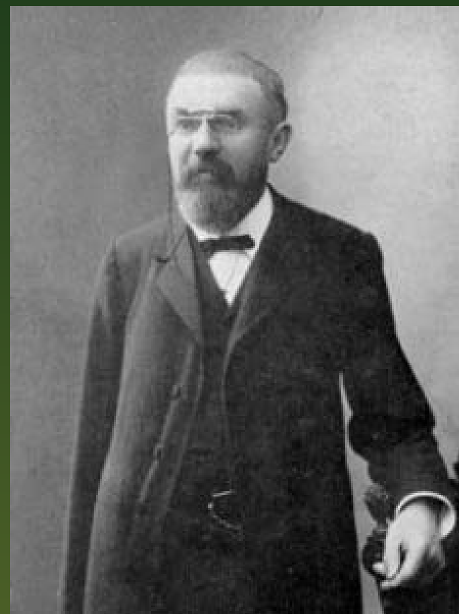
Poincaré（1854－1912）

As to three-dimensional manifolds:

- Poincaré-Hopf theorem holds.
  In three dimensional problems, there are more kind of stationary points that must be considered. If we should give ± sign (integers in general) to each points and sum them all, we get the Euler characteristic of that manifold.
- Euler characteristic of a compact three dimensional manifolds is always zero.
- Are there any other invariants for three-dimensional manifolds?
- The number that Betti（1823－1892） defined is, sure to be an invariant.

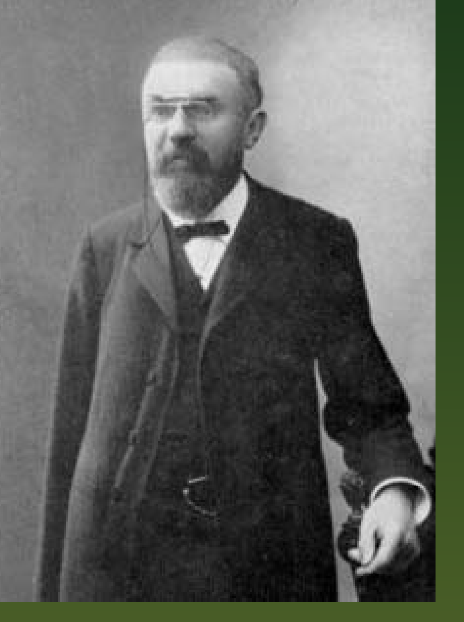


Betti（1823－1892）



Poincaré（1854－1912）

- Poincaré advanced Betti's theory and constructed the Homology theory.
- He conjectured that "any three manifolds whose homology group are isomorphic to the three-dimensional sphere are homeomorphic to three-dimensional sphere" (1900)
- Then Poincaré recognized that the proposition is false, and he constructed a counter-example(1904).
- Poincaré demonstrated that this is in fact a counter-example by defining "fundamental groups".
- And he proposed the problem "Is every simply-connected (trivial fundamental group), compact, three-dimensional manifold homeomorphic to three-dimensional sphere? "



Poincaré（1854－1912）

the counter-example that Poincaré constructed is called Poncare-homology sphere.
Each polytope of the regular 120-polytopes, one of 4-dimensional polyhedrons, is fundamental region of Poincaré-homology sphere.

Poincaré-homology spheres are now recognized to be important in the classification theory of three-dimensional manifolds.

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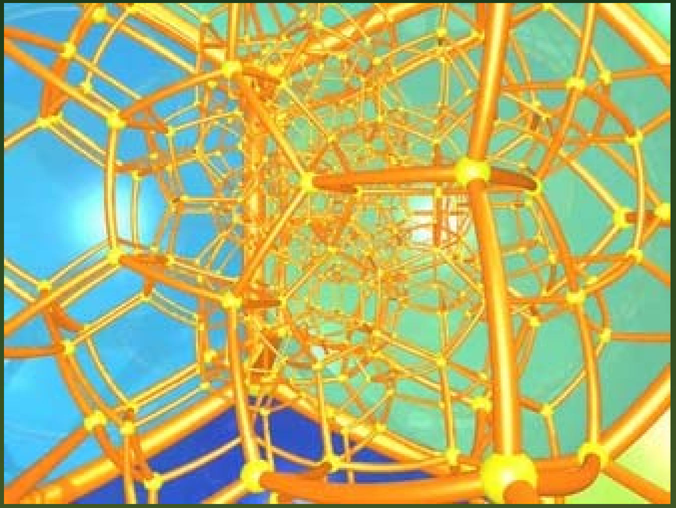
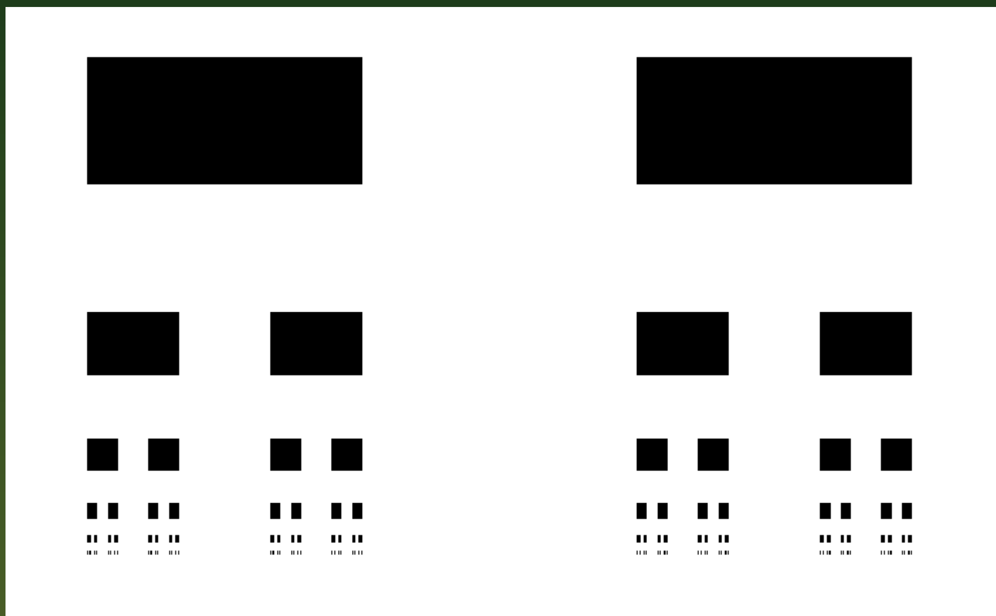


DIMENSIONS
http://www.dimensions-math.org/

Many mathematician, who are contemporary with Poincaré, studied about the figures of spaces.

Cantor(1845－1908) studied comparison of sets and what topology is.



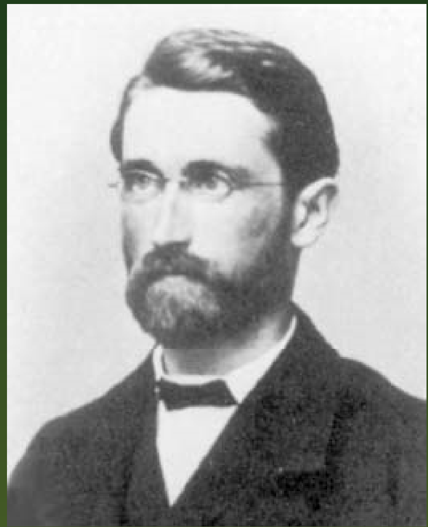
Cantor discovered that there are many kinds of "infinities".

He "demonstrated" that "rational numbers are isolated within real numbers"

The problem "what is number?" also came up.

Peano's definition of natural numbers, and definition of real numbers by Dedekind's cut.

Peano
(1858—1932)



Dedekind
(1831—1916)

Teiji Takagi went to Berlin and Gottingen to study around 1900.

Japanese mathematical teaching and research is based on the tradition of
High level mathematics (Wa-san)

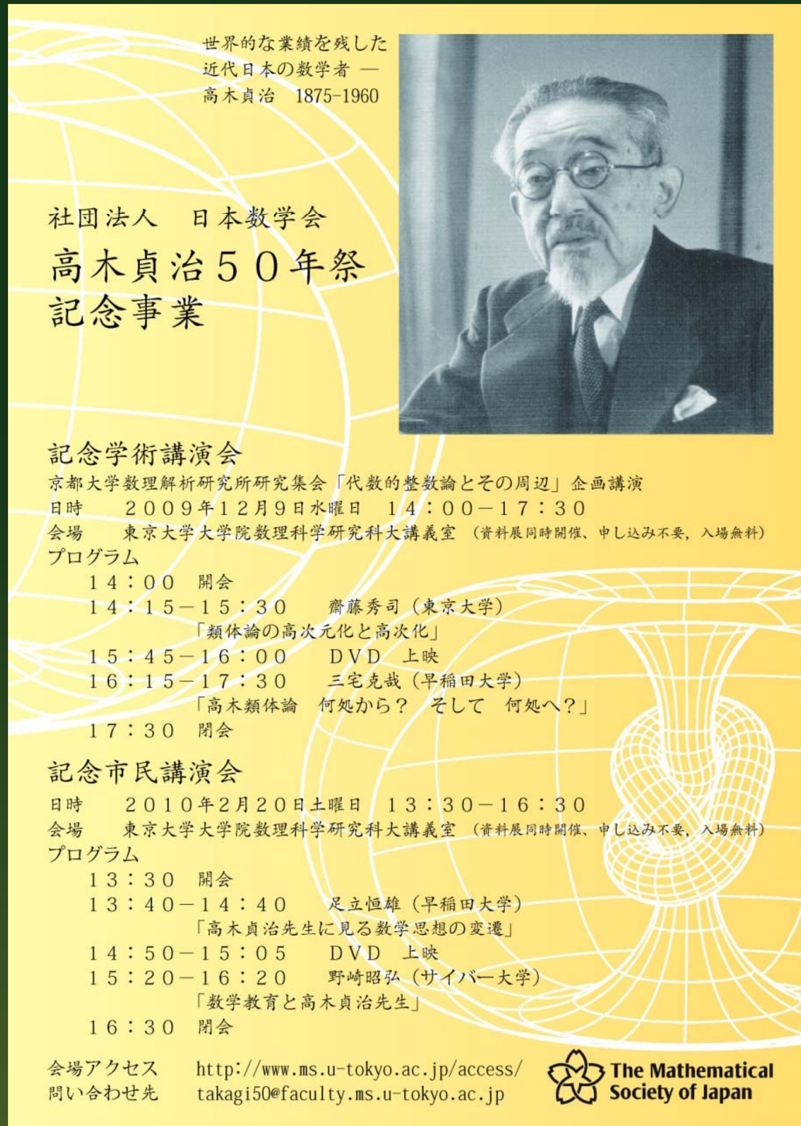
Teiji Takagi teached and studied as the third professor of mathematics at the University of Tokyo.

Teiji Takagi's research on class field theory is highly appreciated and his theory led to the proof of Fermat's last theorem.

Teiji Takagi（1875－1960）
http://www.ms.u-tokyo.ac.jp/summary/gallery.html

In “Kaiseki Gairon”, there are theories on real number that began to be recognized worldwide.

This year is Teiji Takagi’s 50 year anniversary, so there will be a citizen lecture on February 20th at the large lecture room at the department of mathematical study.



http://mathsoc.jp/meeting/takagi50/index.html

By courtesy of Mathematical Society of Japan

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Hopf（1894 － 1971）

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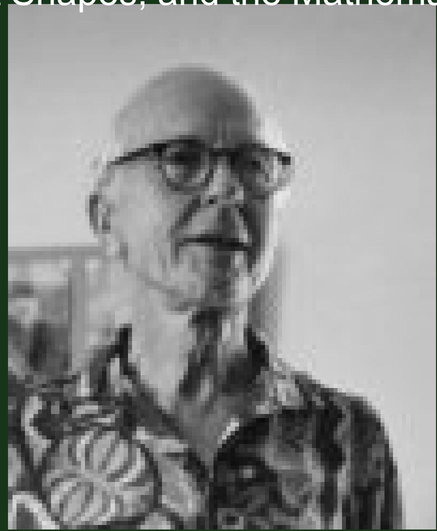

Whitney（1907－1989）
http://www.math.ias.edu/people/past-faculty

de Rham（1903 －1990）

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Thom（1923 － 2002）
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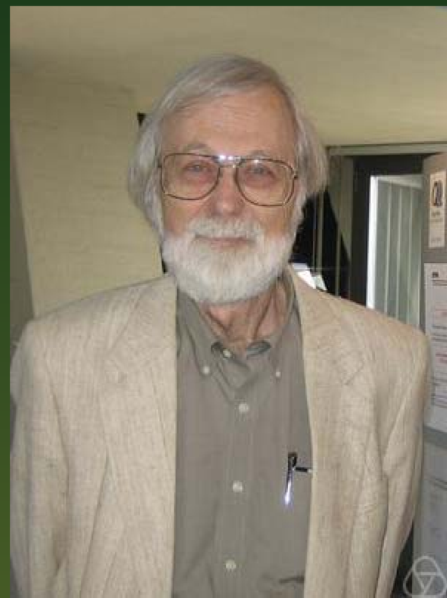

Milner（1931－ ）
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Theory of manifolds are paved by many mathematicians.

- Poincaré conjecture
- Firstly, the same problem with 5 or higher dimension was settled by Smale (1960).
- After that , by Freedman, 4-dimensional Poincaré conjecture is proved(1982).

One of the reasons that the higher-dimensional manifolds are easy to handle is that the intersection of projection of two-dimensional disks and the disk itself cancel.

study of manifolds began by considering functions on them, which are called Morse functions. Then mathematicians tried to simplify their gradient flows, or the maximal gradient lines when Morse functions are seen as height functions.

In other words, they studied manifolds by seeing their shadows (functions) and thinking of the patterns(flows) (solving ordinary differential equations) on them.

For three-dimensional manifolds, it is shown that there are many 3-dimensional manifolds with the same fundamental group but not homeomorphic, which led some people to search for a counter-example of Poincaré conjecture.

Though there are not so many kinds of manifolds as above, more invariants were needed for further classification. It was expected that with such invariants mathematicians would be able to distinguish counter-examples to Poincaré conjecture.

On the other hand, it was proved that, all compact three-dimensional manifolds can be made by connecting finite number of pieces along spheres or two-dimensional tori.

Thurston

Thurston proposed that one of eight geometrical structure can be in each of such pieces, which is called geometrization conjecture (early 1980s). If this is true, Poincaré conjecture is true.

$$\boldsymbol{S}^3,\ \boldsymbol{E}^3,\ \boldsymbol{H}^3$$

$$\boldsymbol{S}^2 \times \boldsymbol{E},\ \boldsymbol{H}^2 \times \boldsymbol{E}$$

$$\mathrm{Nil},\ \mathrm{Sol},\ \widetilde{\boldsymbol{SL_2R}}$$

Thurston formulated the condition that a metric with constant curvature (hyperbolic metric) can exist in the piece. The proof is so massive that it is called “Monster” theorem.

Hamilton came up with an idea making use of Ricci flow, the solution of partial differential equations obtained from deformations of Riemann metrics on manifolds, to deform the metric on manifolds with positive curvature into one with constant curvature.

by courtesy of international mathmaticians union

G. Perelman has discovered certain relations between Ricci flow and the decomposed parts of certain class of manifolds, and therefore showed that for any such decomposed parts, the metric on them always converges into one of eight types.

As a corollary, the 3-dimensional Poincaré conjecture was verified.

‡ 

G.Perelman
by courtesy of Matthias Webe

After this Perelman's work, such a transformation of metric has become popular as a new mathematical method.

For a coordinate near a point on the three-dimensional manifold,
The metric can be represented as $g_{ij} dx_i \otimes dy_j$
While the Ricci curvature can be represented as $R_{ij} dx_i \otimes dy_j$
the following equation describes the variation of metric, the solutions of which are called Ricci flow:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

(This is an equation for 3 x 3 symmetric matrices depending on position and time)

As such, the three-dimensional Poincaré conjecture was demonstrated hearing sounds, observing waves, and measuring how heat is transferred (to solve partial differential equations) on manifolds.

Since manifolds are developed to do calculations on them, methods to solve partial differential equations on manifolds are studied from their early history.

Theories constructed by Atiyah and Singer , Yau, and Donaldson are discovered and the world of manifolds are deepened.

Perelman's proof is also based on these fundamentals.

‡

Copyright: MFO

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by courtesy of
Dr. Simon Donaldson

Summary up to here

- In the study of mechanics (differential equations), the concept of manifolds which is a generalization of curved surfaces are formulated.

- To understand the figure of manifolds, Homology groups and fundamental groups are defined.

- Poincaré proposed a problem that "Are any compact, three-dimensional, simply-connected manifolds homeomorphic to a three-dimensional sphere?" (Poincaré conjecture)
- Thanks to developments of the theory of manifolds, analyzing differential equations on manifolds, Poincaré conjecture was verified.

Since Poincaré conjecture of each dimension is verified, attention is shifted from manifolds themselves to the structure on manifolds.

Though studies on manifolds themselves are based on the research on structures on manifolds, but now more kinds of structures are being studied.

Also for curved surfaces, studies on their complex structure are now going on.

On the other hand, there are a lot of recent studies on areal forms on curved surfaces.

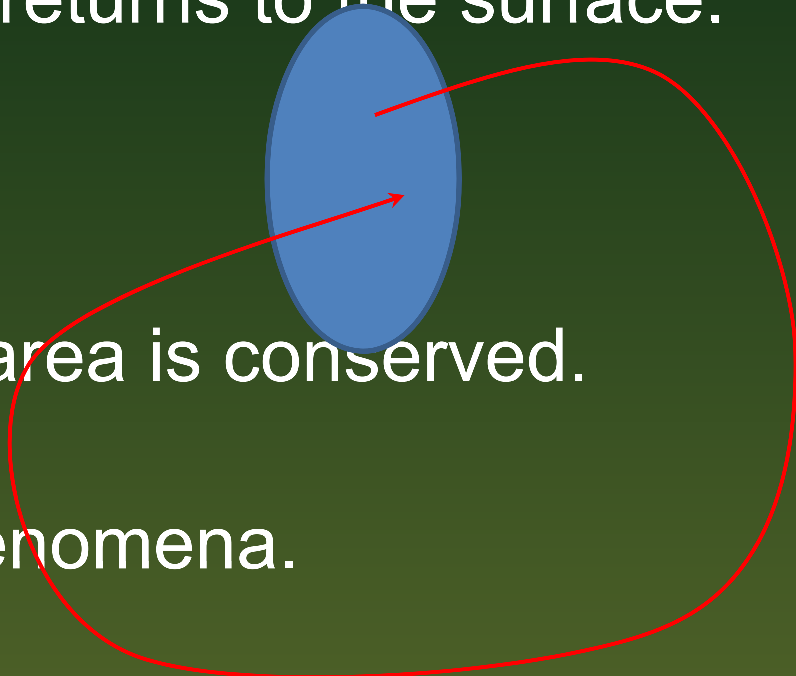
For three-dimensional manifolds, studies on the tangent structure on them are now going on.

For four-dimensional manifolds, studies on complex structures and symplectic structure are now going on.

Studies of areal forms on curved surfaces are related to circular restricted three-body problem.

Solutions to the circular restricted three-body problem are known to be flows on three-dimensional space of state, and the volume is conserved.

For such a flow, if curved surfaces are crossed, the orbit from the surface returns to the surface. This is known to be a recursive map.



For a recursive map, the area is conserved.

Poincaré watched this phenomena.

‡

It is visualized as right figure.

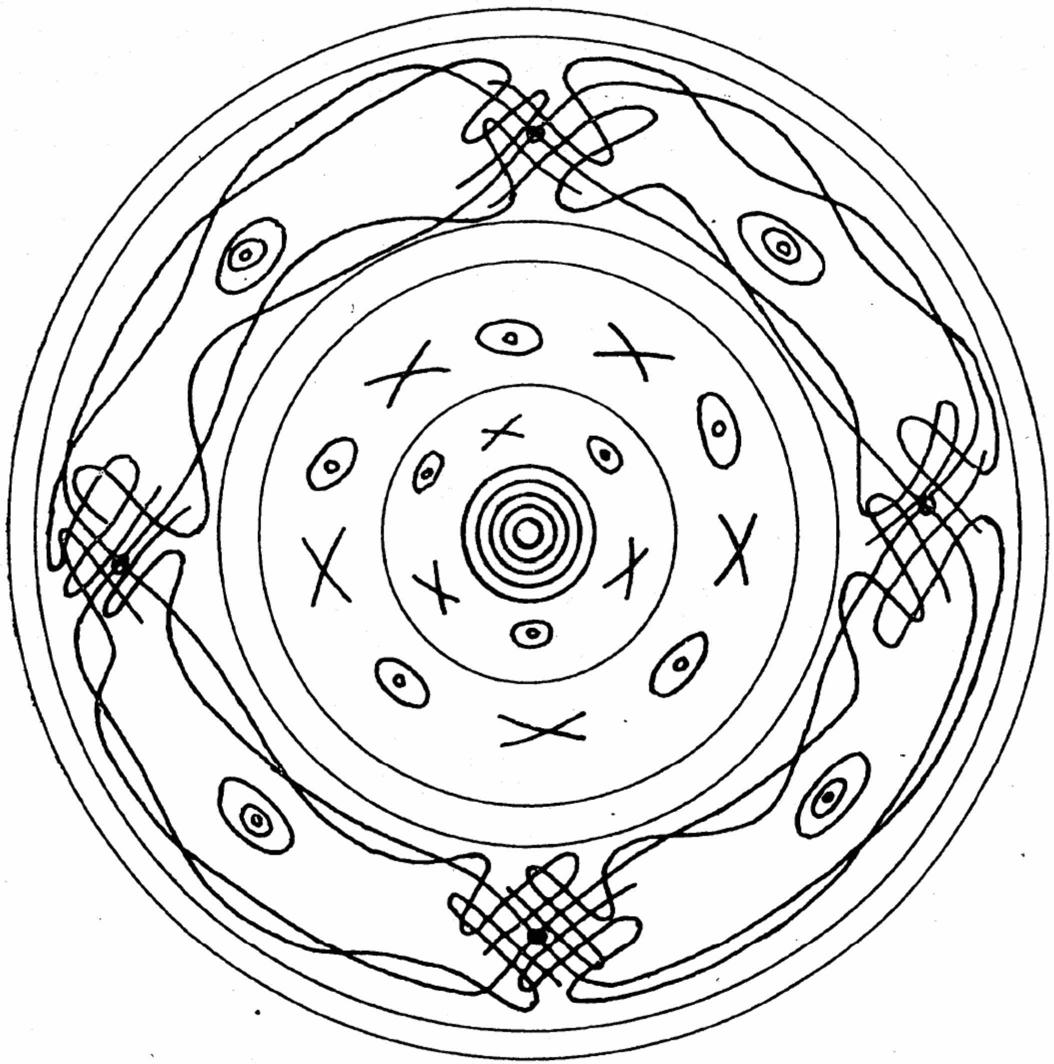


Figure 20.10.

Arnold

Small denominators and problems of stability of motion in classical and celestial mechanics

Figure 6 , Page 93
"Small denominators and problems of stability of motion in classical and celestial mechanics"
Vladimir I Arnol'd, 1963, Russian Mathematical Surveys,18, pp85-191
By courtesy of London Mathematical Sosiety

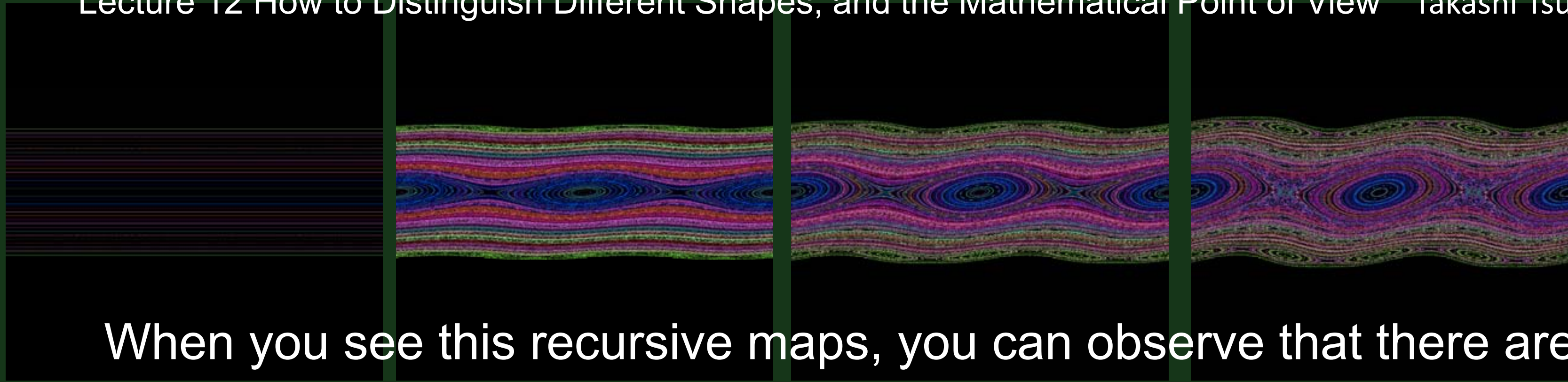
Poincaré, Les Méthodes nouvelles de la mécanique céleste

Que l'on cherche à se représenter la figure formée par ces deux courbes … On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer. Rien n'est plus propre à nous donner une idée de la complication du problème des trois corps et en géneral de tous les problèmes de Dynamique où il n'y a pas d'intégrale uniforme et où les séries de Bohlin sont divergentes.

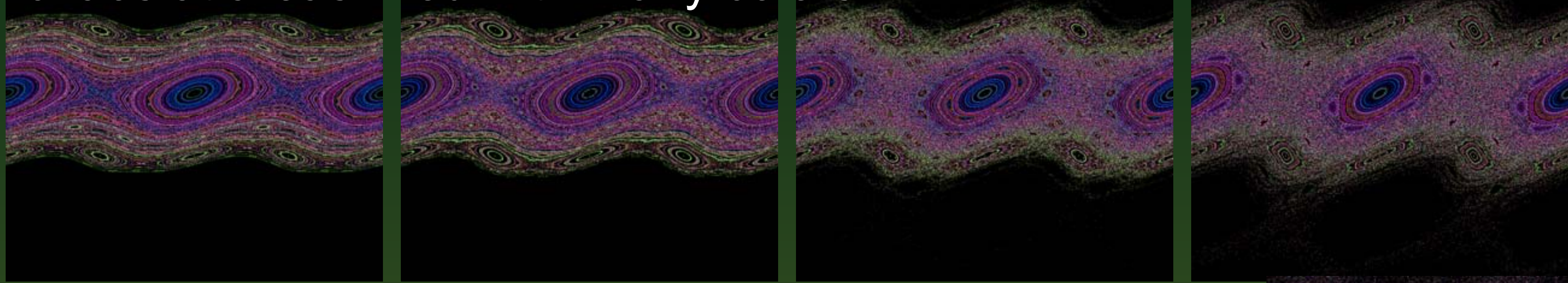
Poincaré had a novel idea on the circular restricted three-body problem.

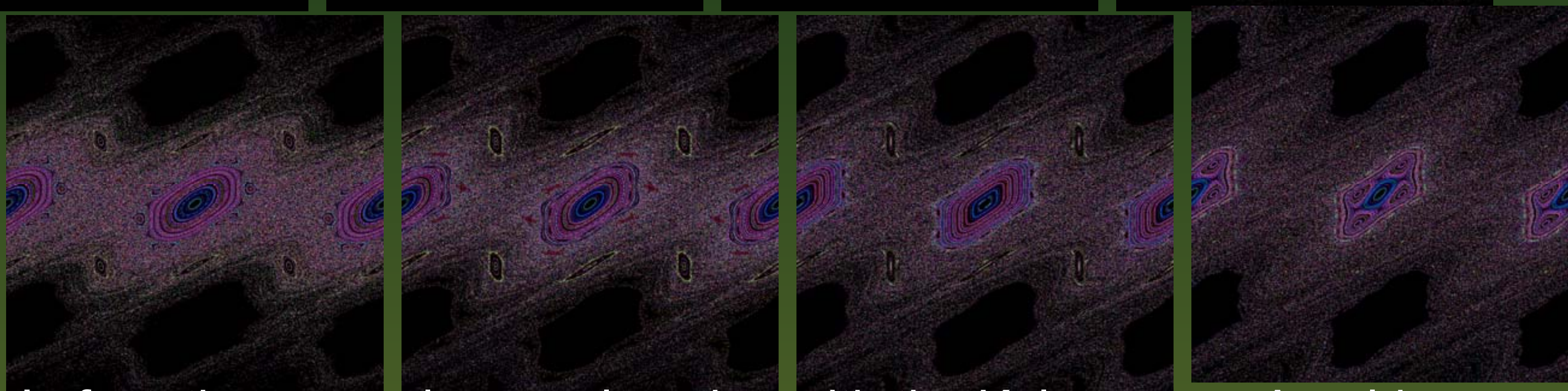
Poincaré came to thinking about observing the difference of recursive maps of flows, varying the parameter of circular restricted three-body problem, namely, the ratio between the mass of Jupiter and the Sun, from zero.

Though I can't explain the details, but if you see the fraction of them, it is almost the same as the following figures.
BASIC source code



When you see this recursive maps, you can observe that there are many circles with the same central points, and that there are signs of chaos at areas filled with many colors.

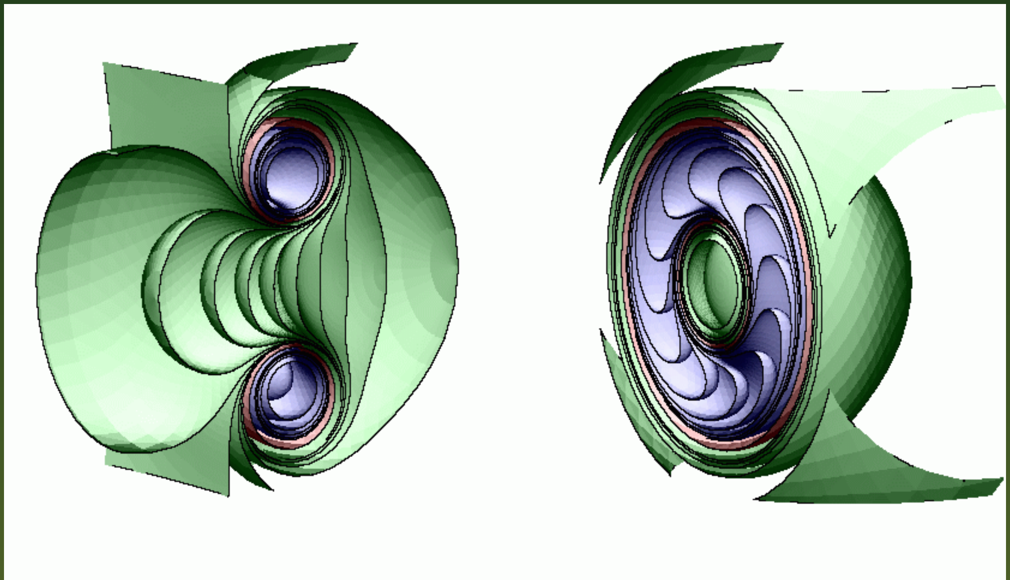


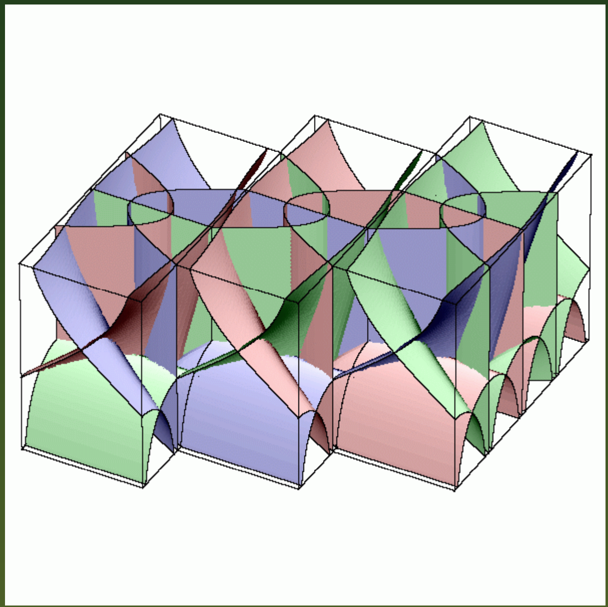


In fact, there are clear explanation with the Kolmogorov-Arnold-Moser theory, and Aubry-Mather theory.

many contemporary mathematical studies are focused on the structure on manifolds.

As an example of structures that can easily be visualized, there are foliated structures, and I am also studying them.
There are animated figures available:
http://faculty.ms.u-tokyo.ac.jp/users/showroom/





- Now, let us return to the beginning of the today's lecture.
- Today I talked about the Poincaré conjecture, but in the beginning the motivation is that we wanted to investigate on the spaces of states. With the knowledge about them, we can restrict the way flows flow and we can get the information.

- Whether curved surfaces are the same or not is, you may think, is distinguished if we carefully observe them.

- But how can we distinguish whether spaces or structures on spaces are the same or not ?

the same

- To determine whether manifolds are the same or not is not easy.
- In principle, the definition of "the same" is needed.
- If two manifolds are homeomorphic(the same figure) that there exists a continuous map from one manifold to another that has the inverse map that is continuous.
- To demonstrate two manifolds are the same, you should either make up such a map, or show that making such a map is possible.
- To demonstrate they are not the same, you should show making such a map is not possible.

The difference: how to distinguish shapes

- To do this, you can use a reduction to absurdity.
- First you can find out some number that is the same if two manifolds are homeomorphic.
- When you find this number above are different for two manifolds, you can conclude these are not the same manifolds.
- Within my talk so far, Euler characteristic and fundamental groups are such numbers.
- Numbers like these are called invariants.

definition of spatial invariants

A certain number is defined for each structure on the space, and if you calculate them and get the same quantity for the same space, irrespective of the structure on them, the number is called spatial invarinats.

Definition of invariants of spatial structure

Suppose one is given a space with certain structure, (then one defines some more structure), and if you calculate certain quantity defined for this structure and always get the same result for spaces with the same structure (and that value is independent of finer structure) we call this quantity the invariants of spaces with structure.

Elegant theorem 1.
Invariants defined using different ways of definition take the same value.
“essential theorem”

Elegant theorem 2.
Two spaces (or structures) with the same invariant is true.
“classification theorem”

Example :linear algebra

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

Let us call that $n \times n$ matrix A, B are “equal” when there exist n-dimensional proper matrices P,Q such that PAQ = B.

If there are nonzero component among the $({}_nC_k)^2$ determinants of submatrices of order k of a matrix and the $({}_nC_{k+1})^2$ submatrices of order k+1 are all zero, the rank of that matrix is defined to be k.

“classification theorem”

That A and B are “the same” and that the rank of A and the rank of B is equal is equal.

For curved surfaces, dividing them with triangles, the value

Number of vertices – number of edges + number of faces

is independent of separation.

This value is called Euler number.

"Morse's theorem"

For functions on curved surfaces, consider critical points composed of minimum points, saddle points, and maximum points and the value

Number of minimum points – number of saddle points + number of maximum points

is equal to Euler characteristic (independent of the function considered)

"Poincaré Hopf theorem"

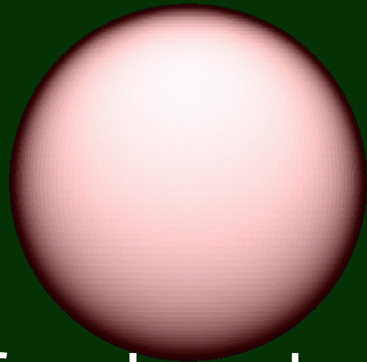
For flows on curved surfaces and if their stationary points are source, saddle, and sink, the value

number of sources – number of saddles + number of sinks

is equal to Euler characteristic

(independent of the flow considered)

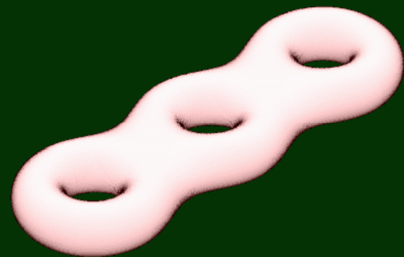
Theorem about Curved surfaces



"Gauss-Bonnet theorem"

The integral of curvature of a closed curved surface in three dimensional Euclidean space is equal to 2π times its Euler characteristic.

"classification theorem"



Closed surfaces in three dimensional Euclidean space are, if their Euler characteristics are equal, homeomorphic.

Theorem of three dimensional manifolds

"Poincaré-Hopf theorem"

If a flow on three-dimensional compact manifolds has finite stationary points, the sum of indices of stationary points is zero.

"Poincaré conjecture"

If the fundamental group of a three-dimensional compact manifold is trivial, the manifold is homeomorphic to a three-dimensional sphere.

How to distinguish different shapes:
Define the invariants and calculate them.

This is not necessarily so difficult.
We simply define some quantity that we think would be useful.
Hard point is that the quantity so defined correctly gives “the same value for the same figures”.

It is not until the time when there are a sufficient ways to distinguish figures that we can tackle the classification problem.

things I sometimes experience as a mathematical researcher

The most is that the invariant I defined with a with pains is, actually, always zero.

This is not a waste. This is a theorem that states some kind of objects have certain nature.

The second kind is that the value I have defined is, actually, already known.

This is a theorem.

But I rarely find the definition of a new and nontrivial invatiant.

References

Teiji Takagi
http://mathsoc.jp/meeting/takagi50/index.html

Milnor：Poincaré conjecture
http://faculty.ms.u-tokyo.ac.jp/users/kokaikoz/milnor-j.pdf

DIMENSIONS
http://www.dimensions-math.org/

The entropy formula for the Ricci flow and its geometric applications, 2002
http://arxiv.org/abs/math.DG/0211159

Ricci flow with surgery on three-manifolds, 2003
http://arxiv.org/abs/math.DG/0303109

Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, 2003
http://arxiv.org/abs/math.DG/0307245