Psychology of Learning and Education

―Understanding and Problem Solving using Visualization―

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Two Meanings of Division

Partitive division：The total number of objects and the number of groups are given, where the unknown factor is the number of objects in each group.

Measurement division：The total number of objects and the number of objects in each group are given, where the unknown factor is the number of groups.

You have 12 cookies and want to divide them equally among 3 people. How many cookies will each person get?

$12 \div 3 = 4$

You have 12 cookies and want to give 4 cookies to each person. How many people can you give cookies to?

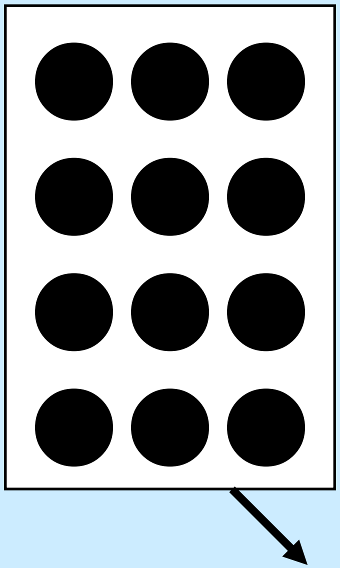
$12 \div 4 = 3$

This is "$3 \times 4 = 12$" to begin with.・・・Division is the inverse operation of multiplication.

Array

3 people

4 cookies per person

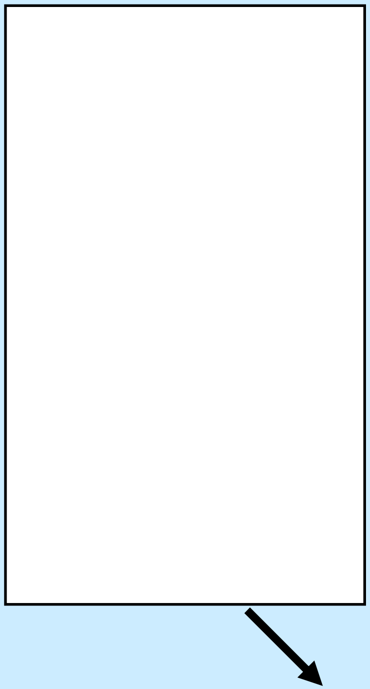


A total of 12 cookies

Area Diagram

The number of units

The number per unit



The total number

Solving the Question of “Cranes and Tortoises” with Area Diagrams

There are a number of cranes and tortoises,16 in all. The number of legs totals 40. How many cranes and tortoises are there?

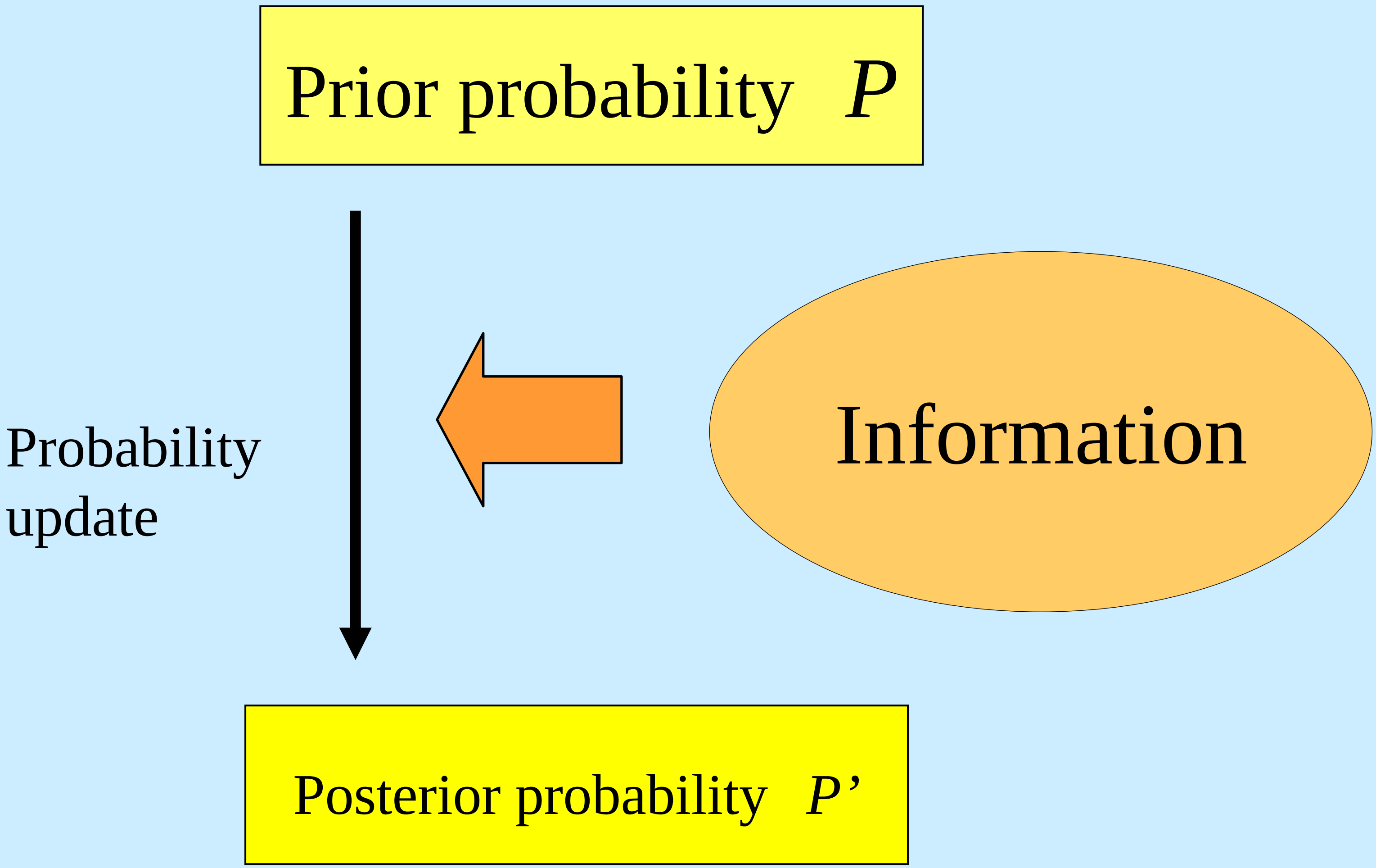
- First, instead of using equations, try to solve the problem by supposing that all of them are cranes.
- Solve the problem using area diagrams.
- Solve the problem using simultaneous equations.
- Compare the solution found using simultaneous equations with that found using area diagrams.

Is this also a question of “cranes and tortoises”?

You want to complete a 42 km marathon in just 6 hours. Assume that you walk at 4 km/h for some distance and run at 12 km/h for the remaining distance. How many hours will you spend walking and running, respectively?

- Solve the problem using area diagrams.
- Solve the problem using simultaneous equations.
- A challenge：Solve the problem by drawing a graph whose horizontal axis represents the time from the start of the race, and whose vertical axis represents the distance.

Today’s main theme : Estimating posterior probability



The Cassette Tape Problem

Tape １： Southern All Stars ／ SMAP
Tape ２： Southern All Stars ／ Southern All Stars

Choose a tape at random

Southern All Stars

The reasoning of people who think “the probability is ½, that the tape chosen is tape 1”

“Both tapes contain footage of the SAS. Since the tape chosen is one of these two tapes, the probability of either tape being that tape is ½.”

“Since both tapes contain footage of the SAS, the fact that the music heard was SAS in no way effects this probability.”

The theoretical answer is １／３.

Repeat the experiment (choosing a tape at random) over and over again.

Use Bayes’ theorem.

The Bayesian Solution to the "Cassette Tape Problem"

$$P' = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1} = \frac{1}{3}$$

Tape 1 SAS Tape 2 SAS

The Infection Problem

Of the people who are infected,
98% display a positive reaction, while 2% display a negative reaction.
Of the people who are not infected,
99% display a negative reaction, while 1% display a positive reaction.

One person in 1,000 is infected → Test drug → Positive

The intuitive reasoning of many people;

"The probability of infection is 98% or more, because the test drug is considered accurate."

The theoretical solution based on Bayes' theorem is 0.089

The Bayesian Solution to the Infection Problem

$$P' = \frac{0.001 \times 0.98}{0.001 \times 0.98 + 0.999 \times 0.01}$$

Infected **Positive** **Not infected** **Negative**

$$= 0.089$$

The number is much smaller than the intuitive solution!
（due to neglect of the prior probability）

The Three Prisoners Problem

Of the three prisoners A, B, and C, one will be pardoned.

A asks the jailor

"B will be executed."

"Since it is certain that at least one of the other two men will be executed, tell me the name of the man, B or C, who is going to be executed."

Conditions

1） The jailor doesn't tell a lie.

2） If both B and C are scheduled to be executed, the jailor will give either name at random.

Bayesian Solution to the Three Prisoners Problem (Original)

$$P' = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1} = \frac{1}{3}$$

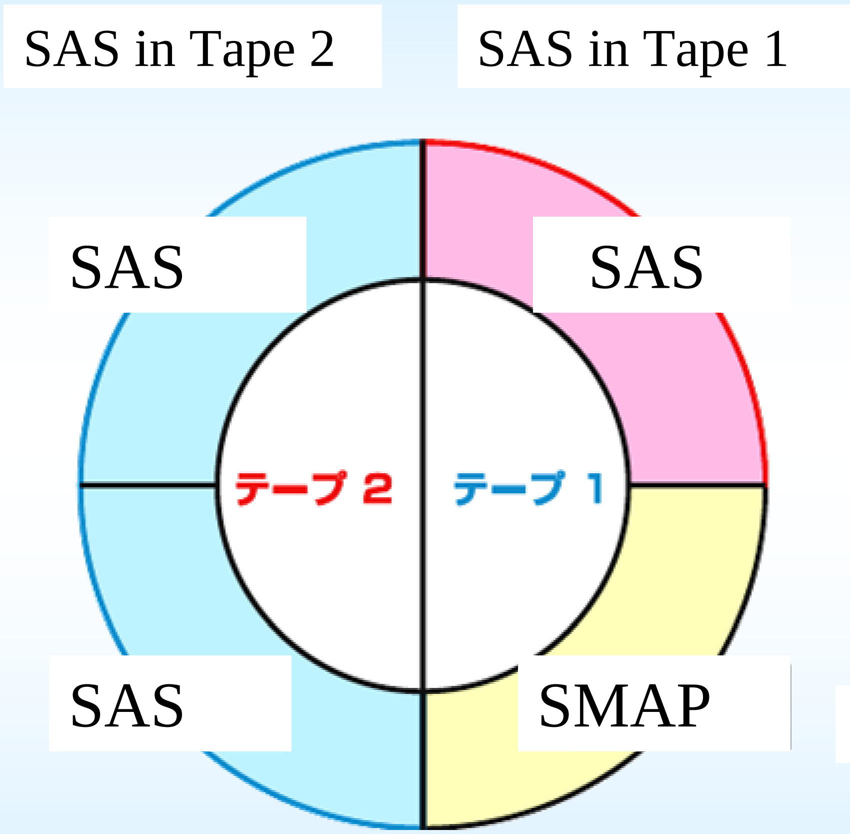
A is pardoned B is pardoned C is pardoned

Bayesian Solution to the Three Prisoners Problem (Modified Version)

$$P' = \frac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 1} = \frac{1}{5}$$

A is pardoned B is pardoned C is pardoned

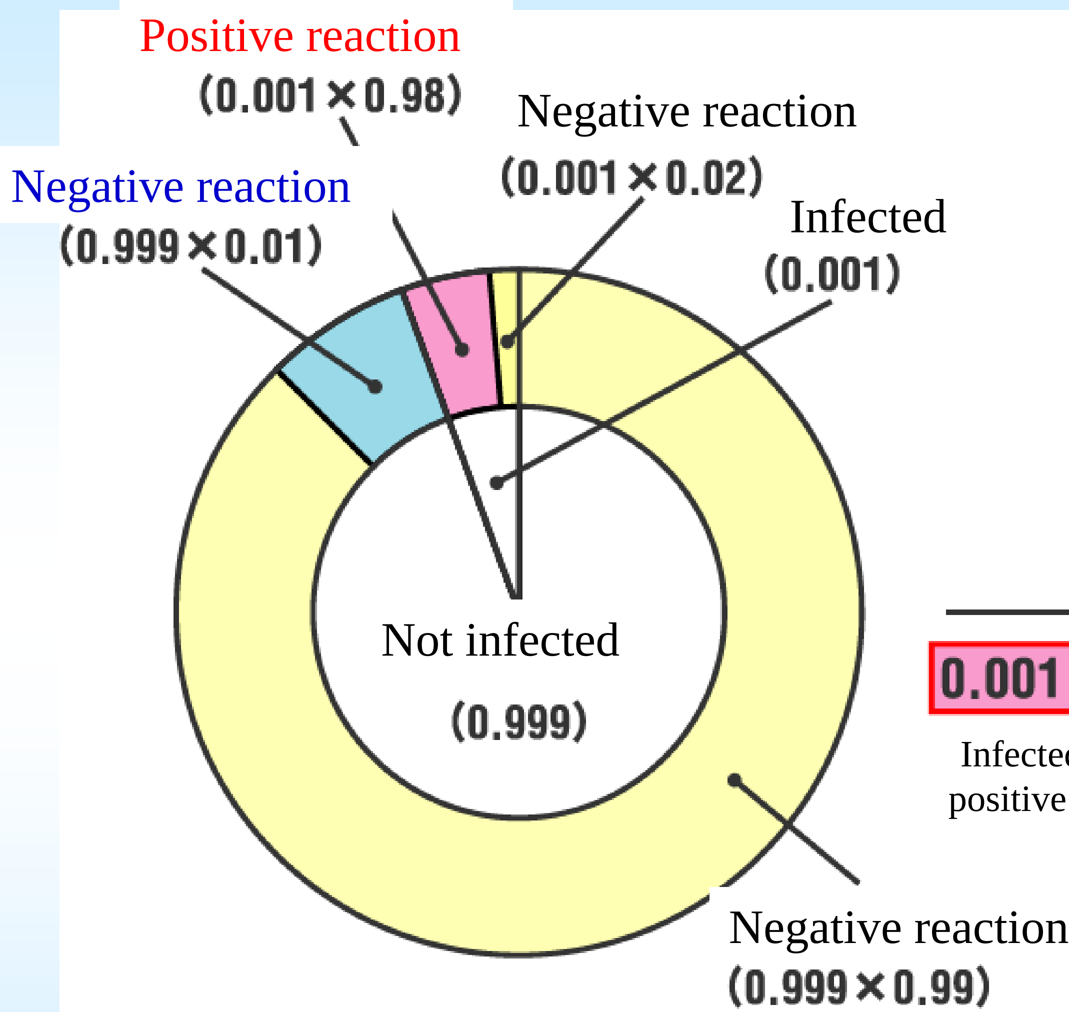
Roulette-wheel Diagram of the “ Cassette Tape Problem”



SAS in Tape 1

$$\frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times 1}=\frac{1}{3}$$

SAS in Tape 1 SAS in Tape 2

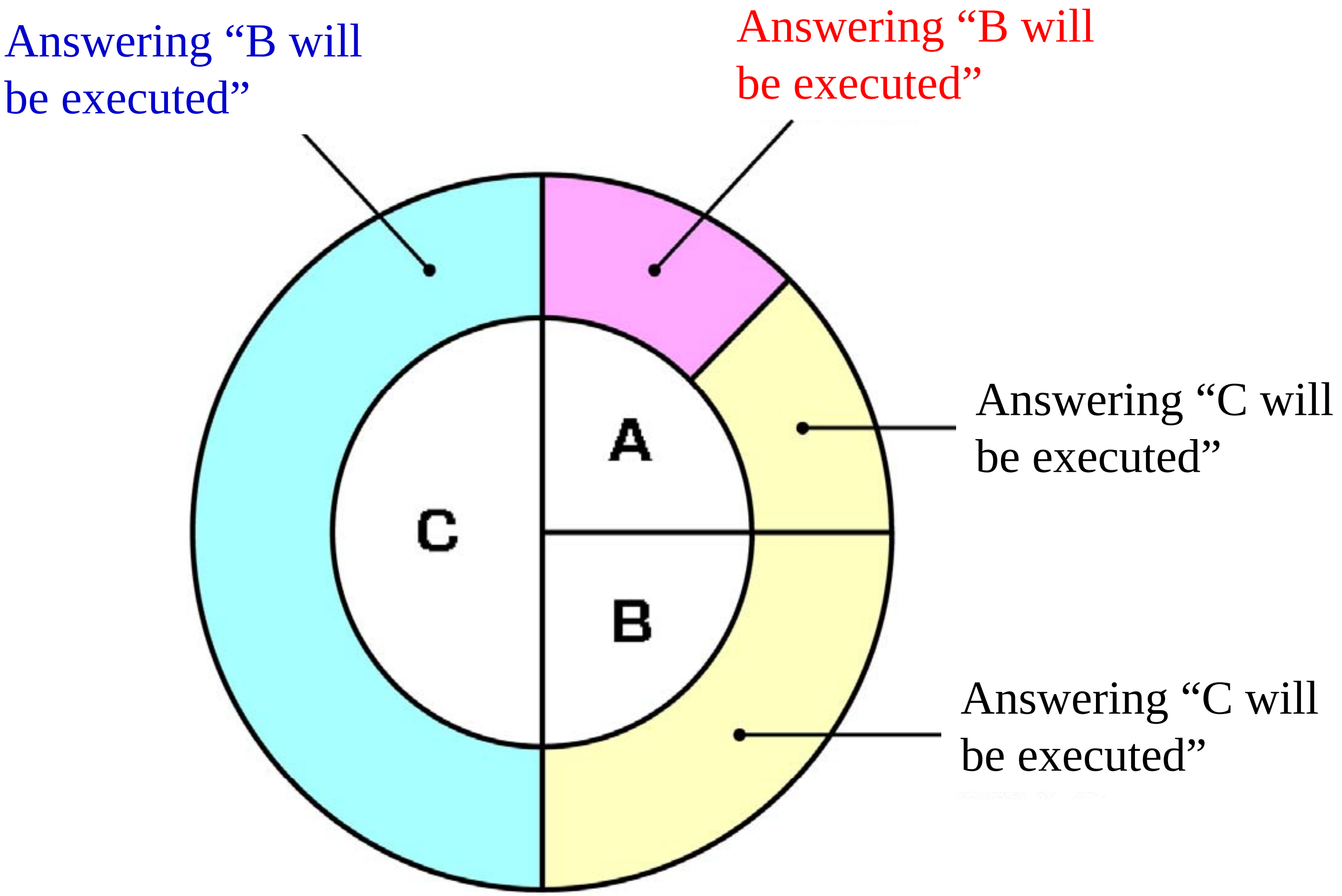


Infected with a positive reaction

$$\frac{0.001 \times 0.98}{0.001 \times 0.98 + 0.999 \times 0.01} = 0.089$$

Infected with a positive reaction | Not infected with a positive reaction

Roulette-wheel Diagram of the “Infection Problem”



Roulette-wheel Diagram of the Modified Three Prisoners Problem

Naive Schema and Heuristics for Judging Probability

$$\text{Probability} = \frac{\text{Certain events}}{\text{All events}}$$

$$\text{Conditional probability} = \frac{\text{Certain events that satisfy conditions}}{\text{All events that satisfy conditions}}$$

How to calculate probability ･･･ by counting and dividing

Conclusions and Development of the Problem of Three Prisoners

Ichikawa, S (1997) *Kangaeru koto no kagaku – Suiron no ninchishinrigaku heno shotai* (The science of thinking – An invitation to the cognitive psychology of reasoning). Chuko shinsho

Ichikawa, S (1998) *Kakuritsu no rikai wo saguru – 3 shujin mondai to sono shuhen* (Exploring an understanding of probability – The problem of three prisoners and the surrounding issues). Kyoritsu Shuppan

Why is the "problem of three prisoners" difficult?

Psychological process in estimating posterior probability

Naive schema and CDH (counting and dividing heuristics)

The characteristics and application of an effective isomorphic schematic representation

Quantitative expression and ease of manipulation

Making the most challenging problems

The problem of the "blindfolded lottery drawing"