Global Focus on Knowledge Lecture Series

Finance and Math

The 9th Lecture

Practical Problems of Finance and

Integral Computations of

Higher Dimensions

How to Compute Integration

Math Application

New Mathematiques

Math Only for Finance

Surely Does Not Exist

There is a need to make math

that yields application.

Mathematical Finance Financial Engineering

Numerical Calculation / Analysis

Practically important.

It is difficult even today when

calculators are well developed!

Traditional methods have their limits.

Computational Finance

Revision (Preparation?) of Integration

Integration of a function $f(x),\ 0 \leqq x \leqq 0,$ $(f(x) \geqq 0,\ 0 \leqq x \leqq 1)$

$$\int_0^1 f(x)dx$$

Area of a figure on xy plane , surrounded by x -axis, y -axis, a graph of $y = f(x)$, a line $x = 1$

Rectangular approximation

$$I_n^{(1)}(f) = \frac{1}{n}f(0) + \frac{1}{n}f(\frac{1}{n}) + \frac{1}{n}f(\frac{2}{n}) + \cdots + \frac{1}{n}f(\frac{n-1}{n})$$

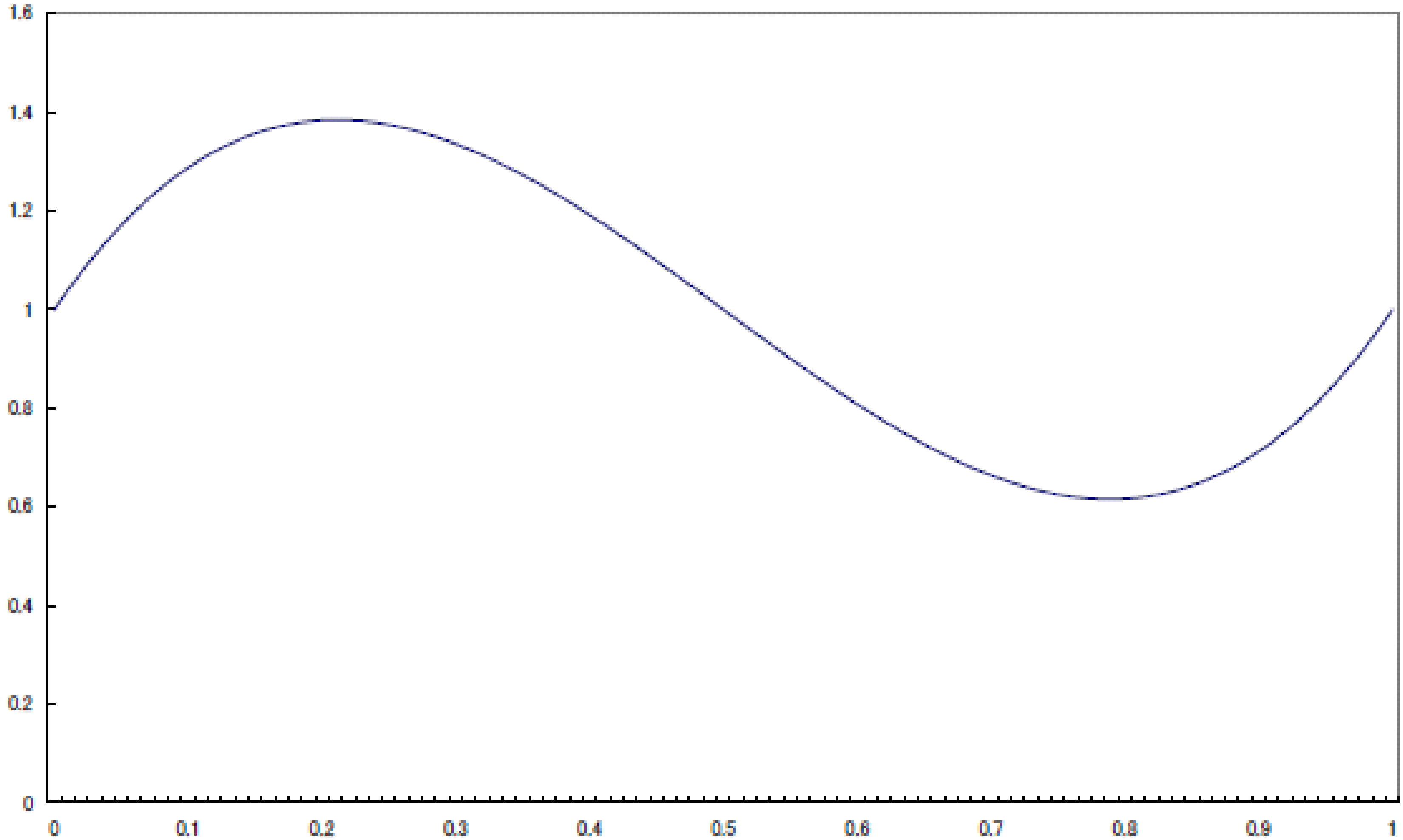
Mathematically, integration means the limit of $I_n^{(1)}$

Trapezoidal approximation

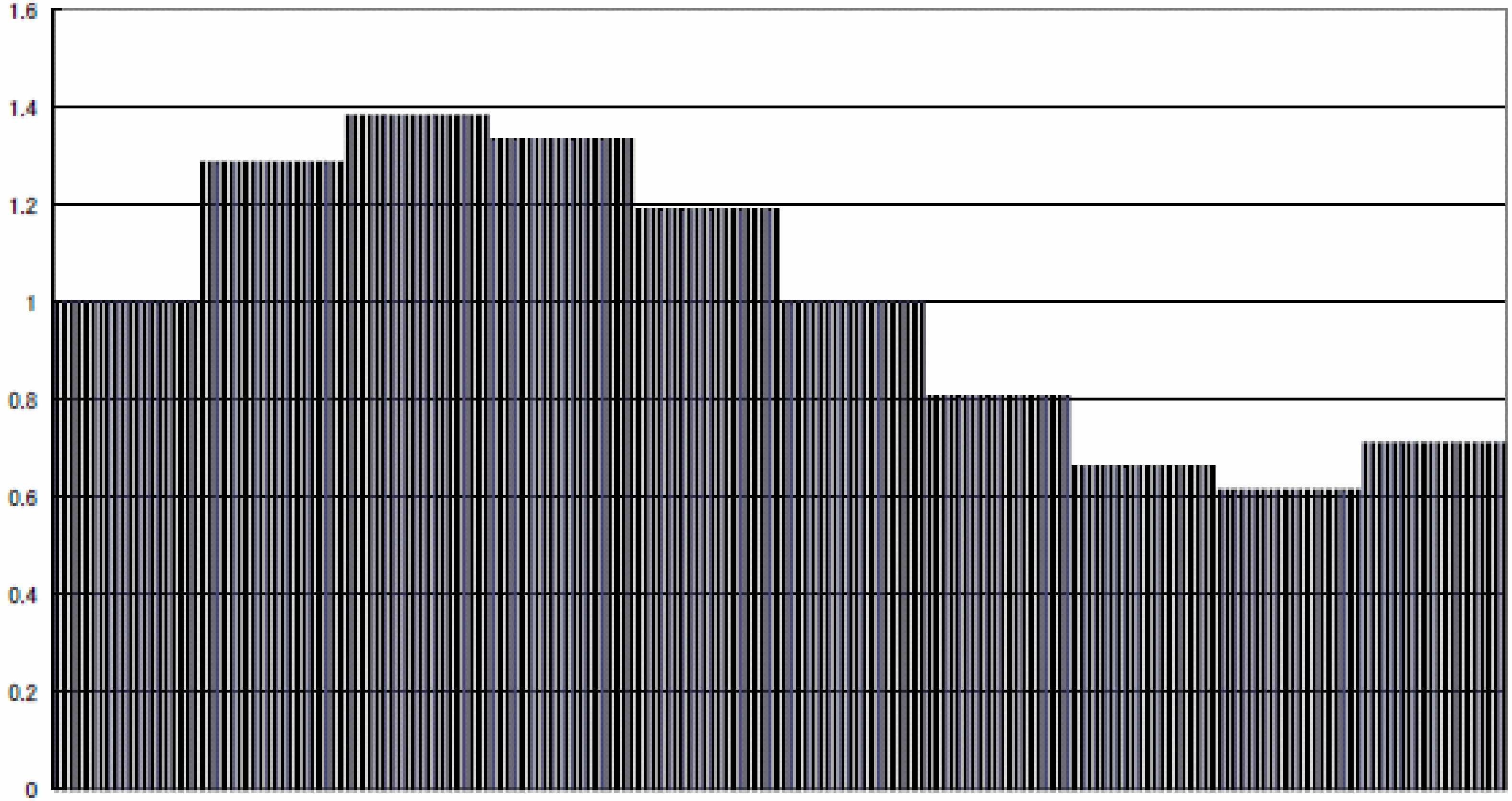
$$I_n^{(1)}(f) = \frac{1}{2n} f(0) + \frac{1}{n} f(\frac{1}{n}) + \frac{1}{n} f(\frac{2}{n}) + \cdots + \frac{1}{n} f(\frac{n-1}{n}) + \frac{1}{2n} f(1)$$

$$\int_0^1 1 \, dx = 1, \qquad \int_0^1 x \, dx = \frac{1}{2},$$

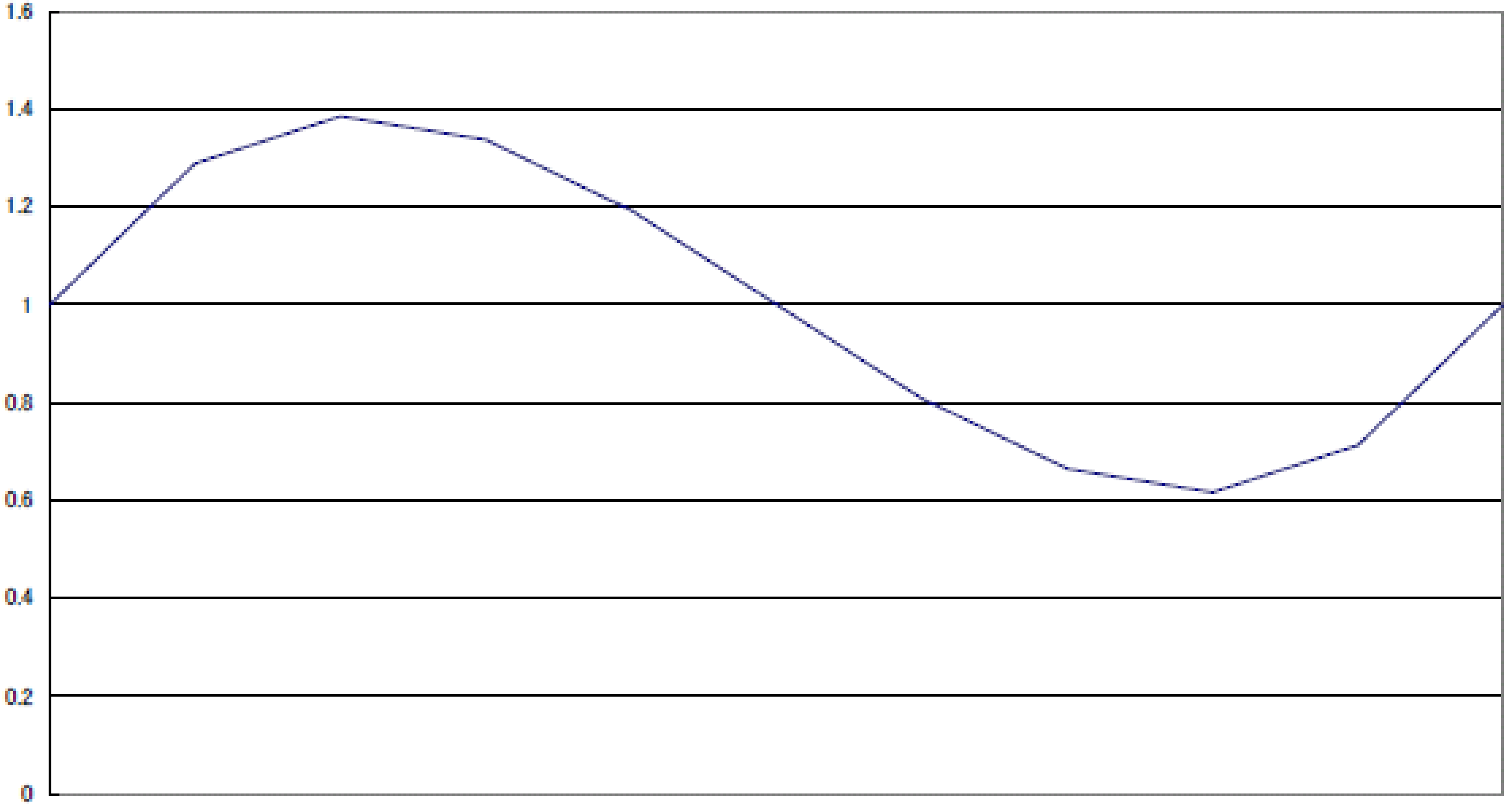
$$\int_0^1 x^m \, dx = \frac{1}{m+1}, \qquad \int_0^1 \frac{4}{1+x^2} \, dx = \pi.$$

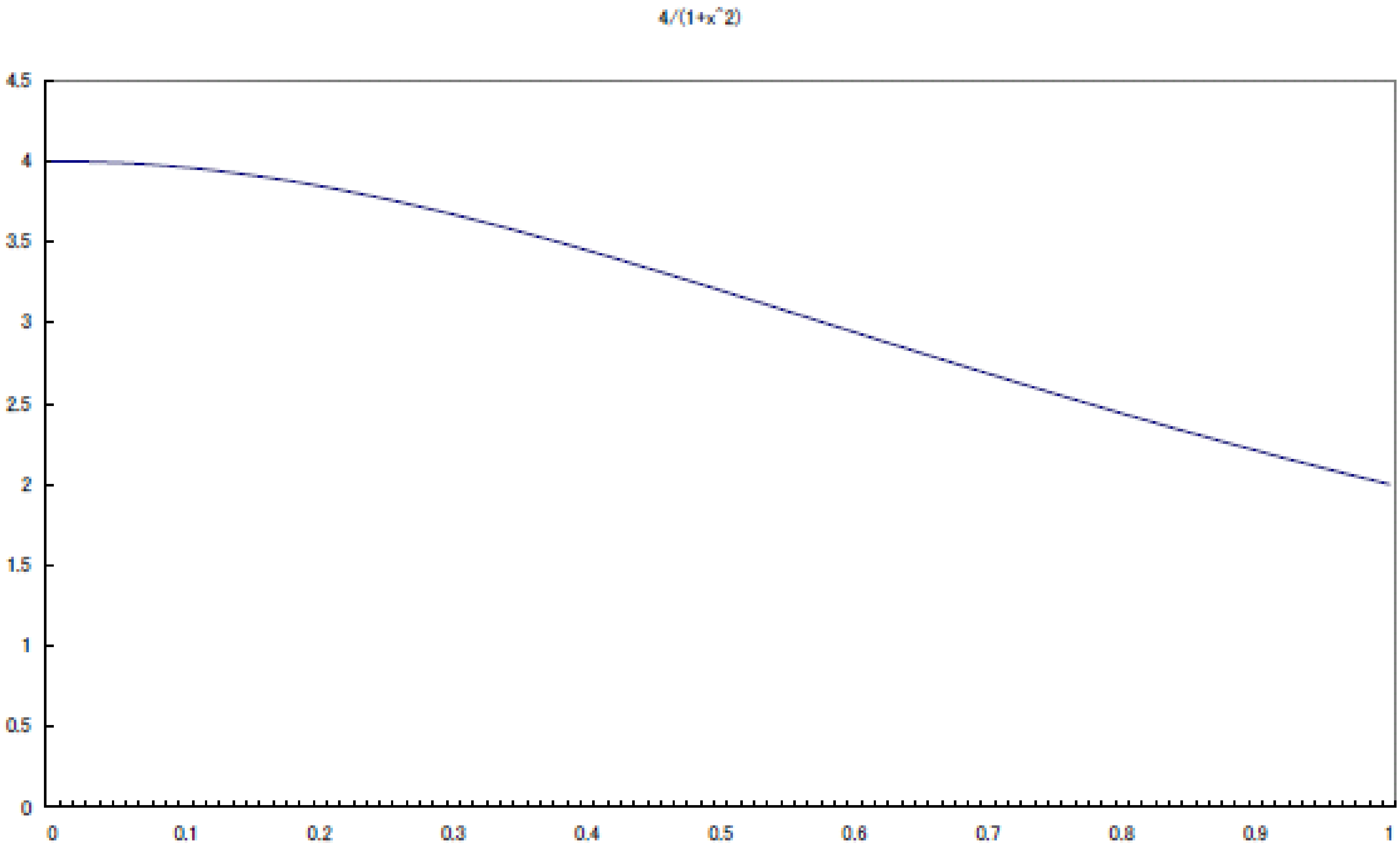

1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Rectangular approximation of integration

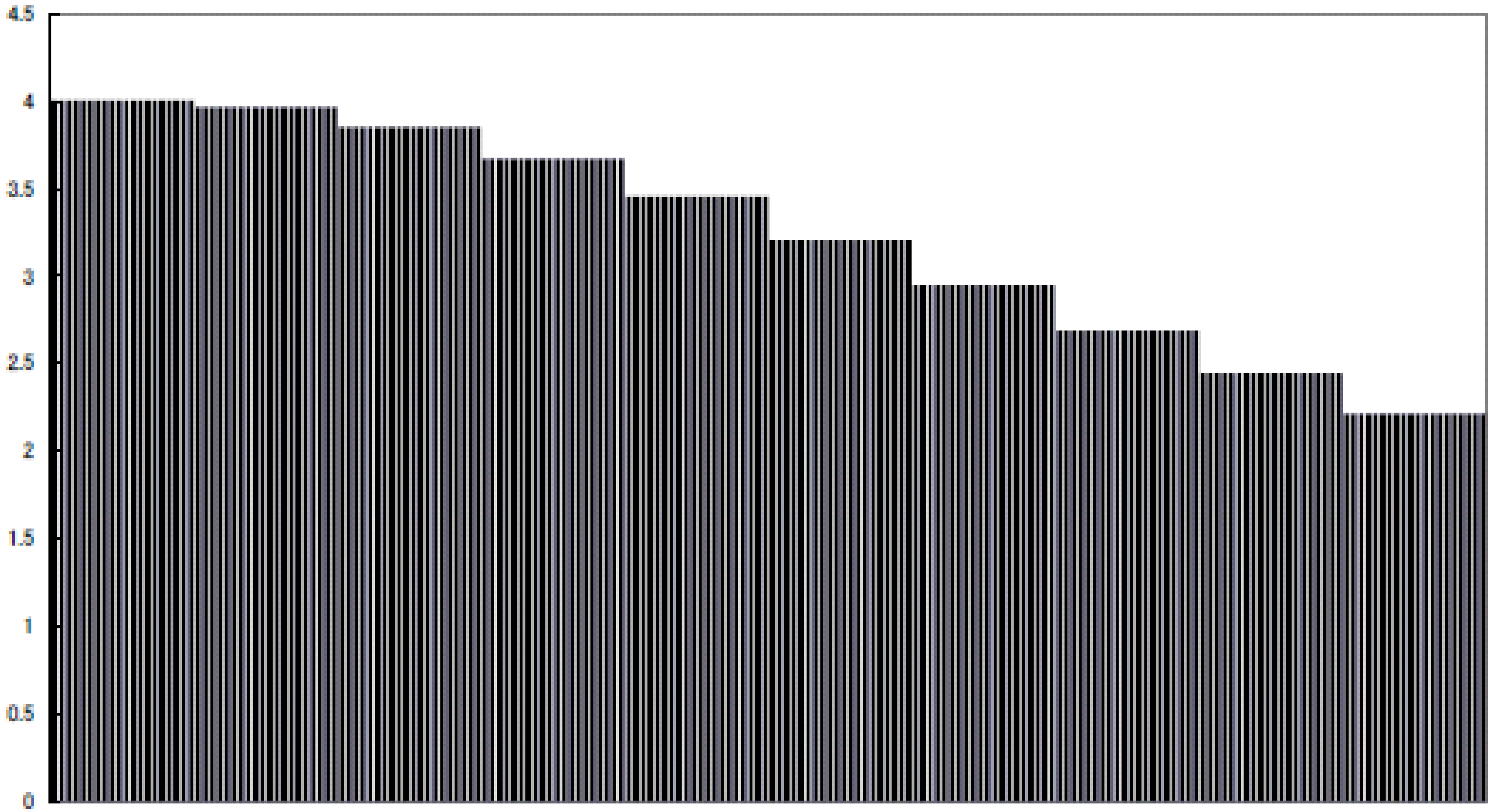


Trapezoidal approximation of integration

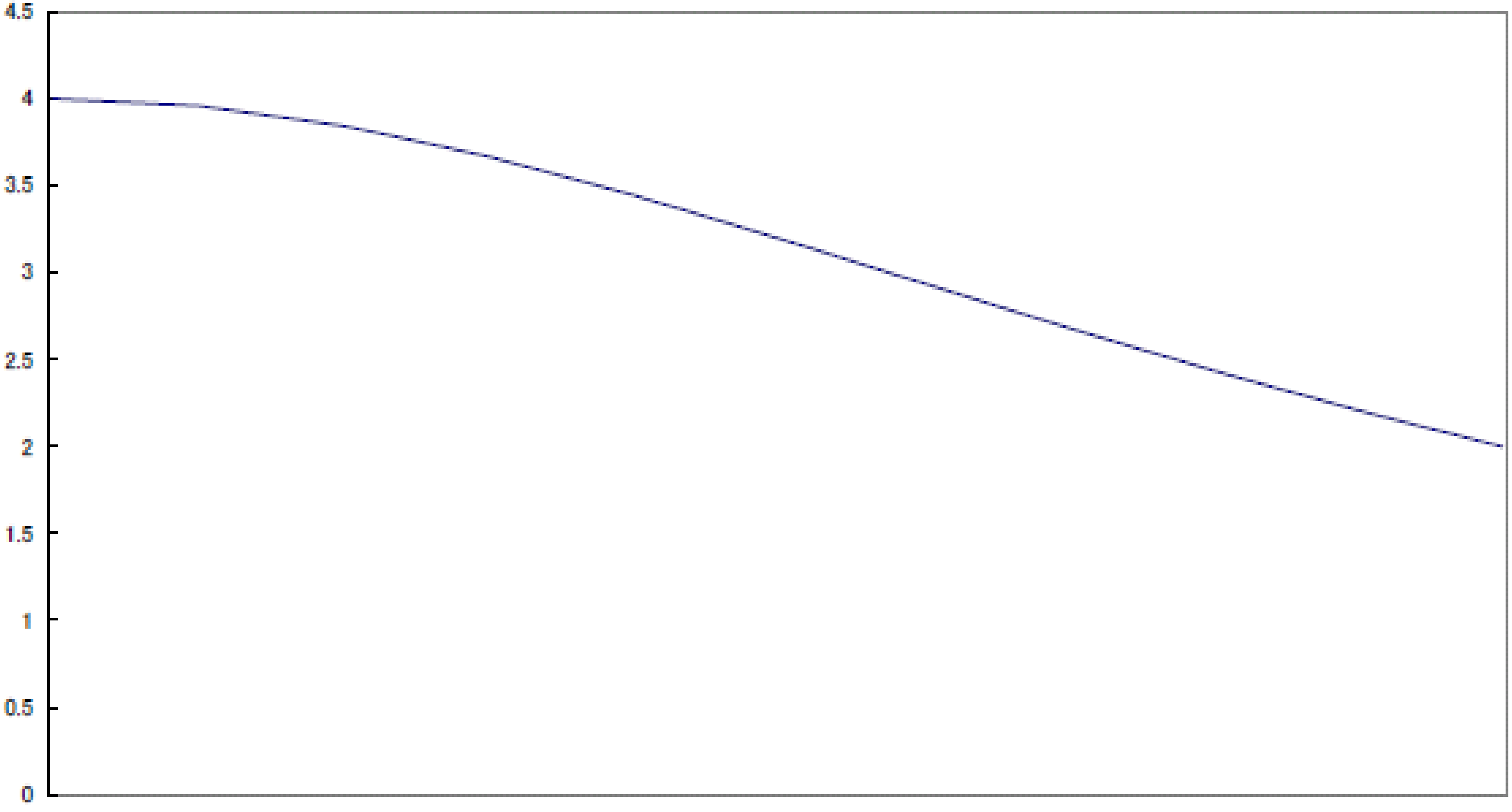



4/(1+x^2)
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Rectangular approximation



Trapezoidal approximation



	rectangle left	trapezoid
100	3.151576	3.141576
200	3.146588	3.141588
300	3.144924	3.141591
400	3.144092	3.141592
500	3.143592	3.141592
600	3.143259	3.141592
700	3.143021	3.141592
800	3.142842	3.141592
900	3.142704	3.141592
1000	3.142592	3.141592

	rectangle left	trapezoid
1000	3.142592	3.141592
2000	3.142093	3.141593
3000	3.141926	3.141593
4000	3.141843	3.141593
5000	3.141793	3.141593
6000	3.141759	3.141593
7000	3.141736	3.141593
8000	3.141718	3.141593
9000	3.141704	3.141593
10000	3.141693	3.141593

Integration of Multivariable Function (Multiple Integration)

Multiple Integration of 2-variables Function $f(x,y),\ 0 \leqq x, y \leqq 1$

$$\int_{0\leqq x,y\leqq 1} f(x,y)dxdy = \int_0^1 (\int_0^1 f(x,y)dy)dx$$

$$I_n^{(2)}(f)$$

$$\begin{aligned}
&= \frac{1}{n^2}\{f(0,0) + f(0,\frac{1}{n}) + f(0,\frac{2}{n}) + \cdots + f(0,\frac{n-1}{n}) \\
&\quad + f(\frac{1}{n},0) + f(\frac{1}{n},\frac{1}{n}) + f(\frac{1}{n},\frac{2}{n}) + \cdots + f(\frac{1}{n},\frac{n-1}{n}) \\
&\quad + f(\frac{2}{n},0) + f(\frac{2}{n},\frac{1}{n}) + f(\frac{2}{n},\frac{2}{n}) + \cdots + f(\frac{2}{n},\frac{n-1}{n}) \\
&\quad + f(\frac{n-1}{n},0) + f(\frac{n-1}{n},\frac{1}{n}) + f(\frac{n-1}{n},\frac{2}{n}) + \cdots + f(\frac{n-1}{n},\frac{n-1}{n})\}
\end{aligned}$$

$I_n^{(2)}$ has n^2 terms.

If $f(x,y) = g(x) + h(y)$

$$I_n^{(2)}(f) = I_n^{(1)}(g) + I_n^{(1)}(h)$$

Left side is a term of n^2, right side is a term of $2n$

Amount of calculation is overwhelmingly small.

In the same way, integration of 3-variables function $f(x_1, x_2, x_3),\ 0 \leqq x_1, x_2, x_3 \leqq 1$ is

$$\int_{0 \leqq x_1, x_2, x_3 \leqq 1} f(x_1, x_2, x_3) dx_1 dx_2 dx_3 = \int_0^1 (\int_0^1 (\int_0^1 f(x_1, x_2, x_3) dx_3) dx_2) dx_1$$

Approximation

$$I_n^{(3)} = \frac{1}{n^3}\{f(0,0,0) + \cdots + f(\frac{n-1}{n}, \frac{n-1}{n}, \frac{n-1}{n})\}$$ (term of n^3)

The more variables, the more amount of calculation, so approximation of integration becomes more difficult.

Integration of 360-variables Function

The worth of MBS (Mortgage Backed Security)

Securities based on housing loan

Debt guarantee from the government: No risk of non-performing loan

A borrower of the loan has the right to repay ahead of schedule.

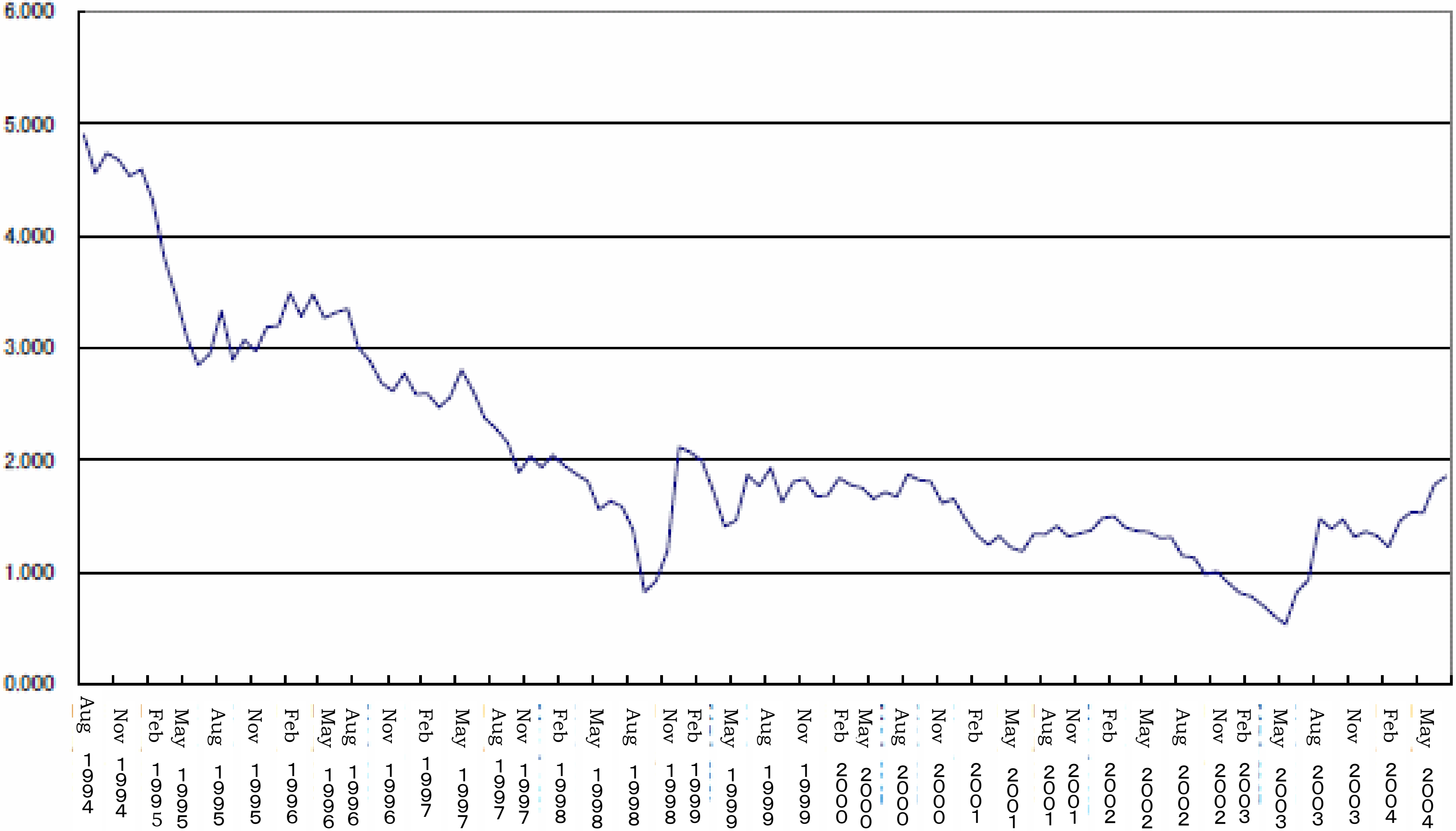
Risk of a prompt return: One might switch to a loan with less interest.

Stochastic process model for interest

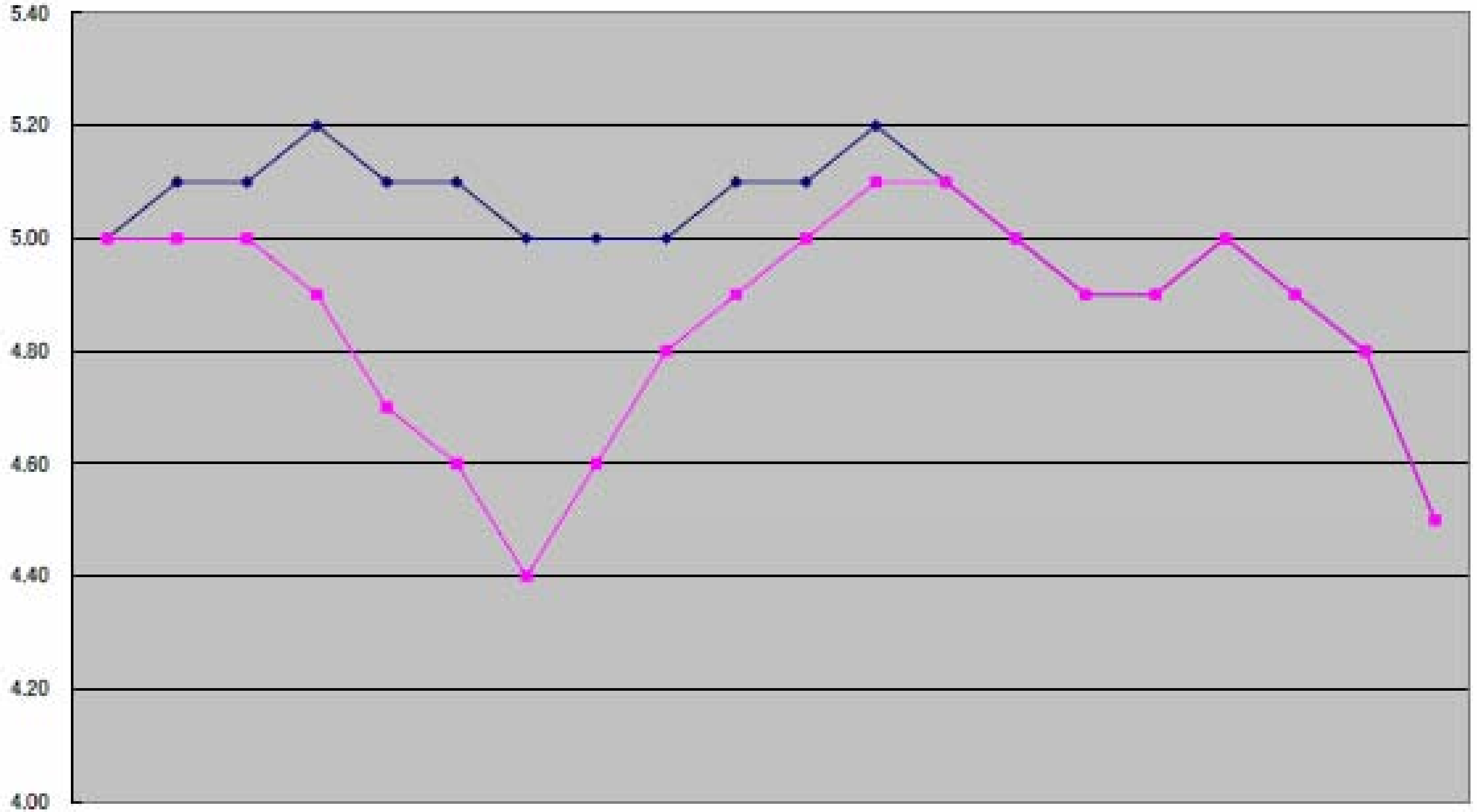
Model for a degree of responses toward interest change

The longest repayment period is 30 years. = 360 months

Yield Rate of Long-term Government Securities



Transition of Interest



Mathematical Problem

Calculation of multiple integration of 360-variables function

$f(x_1, x_2, \ldots, x_{360}),\ 0 \leqq x_1, x_2, \ldots, x_{360} \leqq 1$

$$\int \cdots \int_{0 \leqq x_1, x_2, \ldots, x_{360} \leqq 1} f(x_1, x_2, \ldots, x_{360})\, dx_1\, dx_2 \cdots dx_{360}$$

Quadrature method is to be used.

Divide areas each variables move in into 10 sections and calculate.

10^{360} computations are needed.

10^{30} computations in a second : 10^{330} seconds $>$ 10^{320} years

Method Using Random Numbers

To make it easier, let us consider integration of 2-variables function $f(x,y),\ 0 \leqq x, y \leqq 1$
(The idea here can be applied to integration of 360-variables function.)

◇ Monte Carlo Method

Uniform random numbers X (real numbers between 0 and 1)

Uniform random numbers: Realization value of chance variable whose probability to be $0 \leqq X \leqq a\ (0 \leqq a \leqq 1)$ is a

$X_1, X_2, X_3, \ldots$ are independent uniform random numbers .

Law of great numbers

$$M_n = \frac{1}{n}\{f(X_1, X_2) + f(X_3, X_4) + \cdots + f(X_{2n-1}, X_{2n})\}$$

$$\to I = \int\int_{0 \leqq x, y \leqq 1} f(x,y)\,dx\,dy, \qquad n \to \infty$$

Actual appearance of uniform random numbers:

pseudorandom numbers $\Longrightarrow$ Makoto Matsumoto (the U. of Hiroshima)

Loosely speaking, the error is $|I - M_n| \sim n^{-1/2}$

(Accurately, $n^{-1/2} \log\log n$)

Error around 10^{-3} : About $n = 10^6$ calculations are needed.

Error around 10^{-5} : About 10^{10} calculations are needed.

A great decrease in amount of calculations.

Isn't there a point sequence with less bias?

◇ Halton Sequence notation using binary or ternary system

decimal system	binary system	ternary system
1	1	1
2	10	2
3	11	10
4	100	11
5	101	12
6	110	20
7	111	21
8	1000	22
9	1001	100
10	1010	101

Calculation of a Number n Using Binary Notation

n divided by $2 = c_{n1}$, remainder a_{n1} c_{n1} divided by $2 = c_{n2}$, remainder a_{n2}

c_{n2} divided by $2 = c_{n3}$, remainder a_{n3}

Set following sequence $a_{n1}, c_{n1}, a_{n2}, c_{n2}, a_{n3}, c_{n3}, \cdots$ in the same way

$a_{n1}, a_{n2}, a_{n3}, \cdots$ are 0 or 1, and

$$n = a_{n1}2^0 + a_{n2}2^1 + a_{n3}2^2 + \cdots$$

When n is written in binary notation, a_{n1} is first digit, and a_{n2} is tenth digit

$$\phi_2(n) = \frac{a_{n1}}{2} + \frac{a_{n2}}{2^2} + \frac{a_{n3}}{2^3} + \cdots$$

Determined as above, a function ϕ_2 that correspond natural number n to a fraction (binary number) $\phi_2(n)$, and $0 < \phi_2(n) < 1$

decimal system	binary system	$\phi_2(n)$
0	0	0
1	1	$1/2$
2	10	$0 + 1/4$
3	11	$1/2 + 1/4$
4	100	$0 + 0 + 1/8$
5	101	$1/2 + 0 + 1/8$
6	110	$0 + 1/4 + 1/8$
7	111	$1/2 + 1/4 + 1/8$
8	1000	$0 + 0 + 0 + 1/16$
9	1001	$1/2 + 0 + 0 + 1/16$
10	1010	$0 + 1/4 + 0 + 1/16$

Generally, when p is a prime number, and n is developed to p-adic number,

$$n = b_{n1}p^0 + b_{n2}p^1 + b_{n2}p^1 + \cdots$$

Only 1 $b_{n1}, b_{n2}, b_{n3}, \ldots = 0, 1, 2, \ldots, p-1,$ that satisfies above is determined.

When

$$\phi_p(n) = \frac{b_{n1}}{p} + \frac{b_{n2}}{p^2} + \frac{b_{n3}}{p^3} + \cdots$$

a function ϕ_p that corresponds a fraction satisfying $0 < \phi_p(n) < 1$ to a natural number n is determined.

Table of $(\phi_2(n), \phi_3(n))$ When $1 \leqq n \leqq 9$

n	$\phi_2(n)$	$\phi_3(n)$
1	$1/2$	$1/3$
2	$1/4$	$2/3$
3	$3/4$	$1/9$
4	$1/8$	$4/9$
5	$5/8$	$7/9$
6	$3/8$	$2/9$
7	$7/8$	$5/9$
8	$1/16$	$8/9$
9	$9/16$	$1/27$
10	$5/16$	$10/27$

For a vector sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ inside a square $[0,1]^2$, a figure B in the square, and natural number $N \geqq 1$,

$$D_N(B; \{(x_n, y_n)\}_{n=1}^{\infty})$$

$$= |((x_n, y_n) \text{ in figure } B, \text{ number of } 1 \leqq n \leqq N) - N \times (\text{area of } B)|$$

is determined. (Discrepancy)

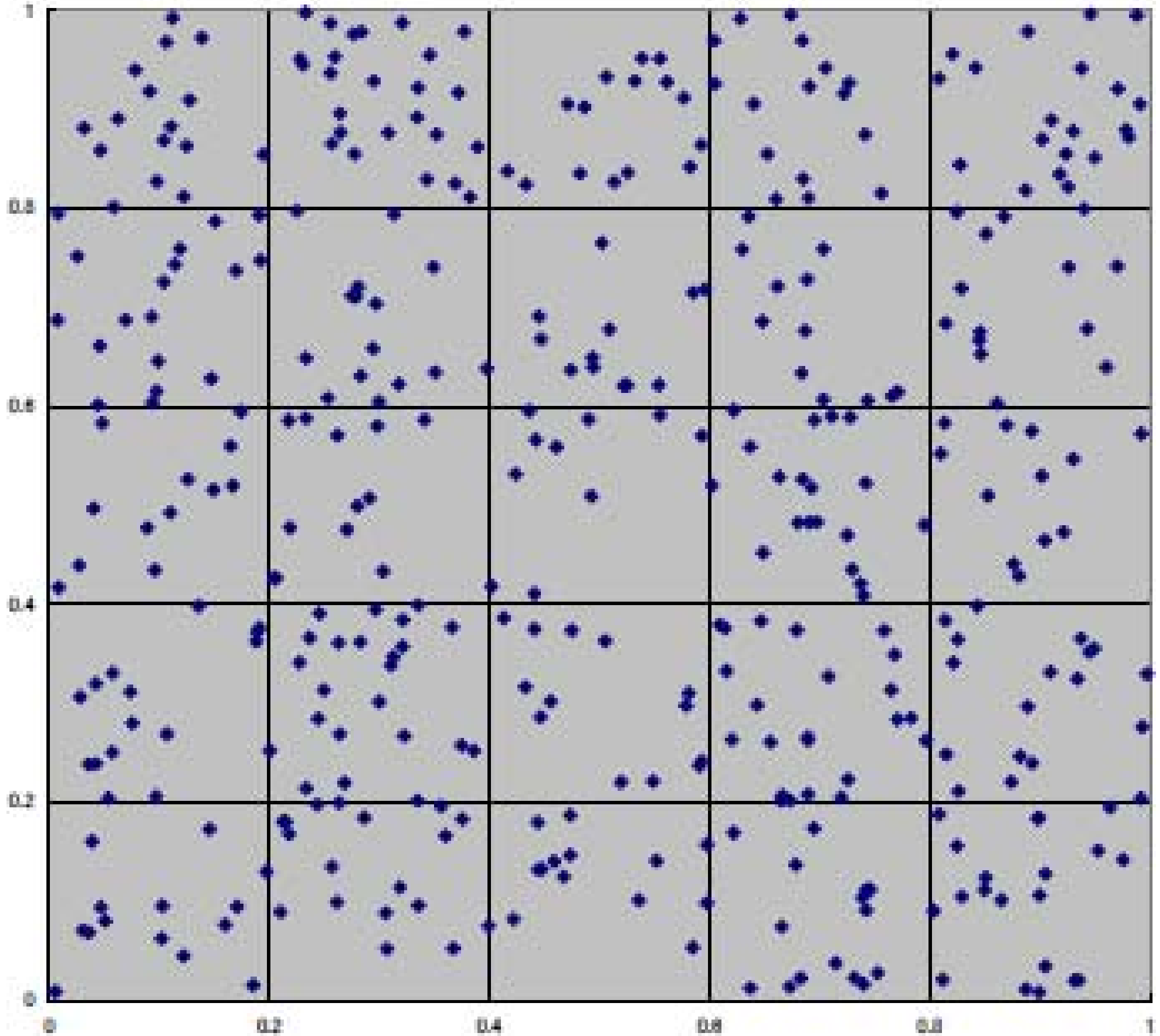
Theorem 1 Suppose p, q are different prime numbers.

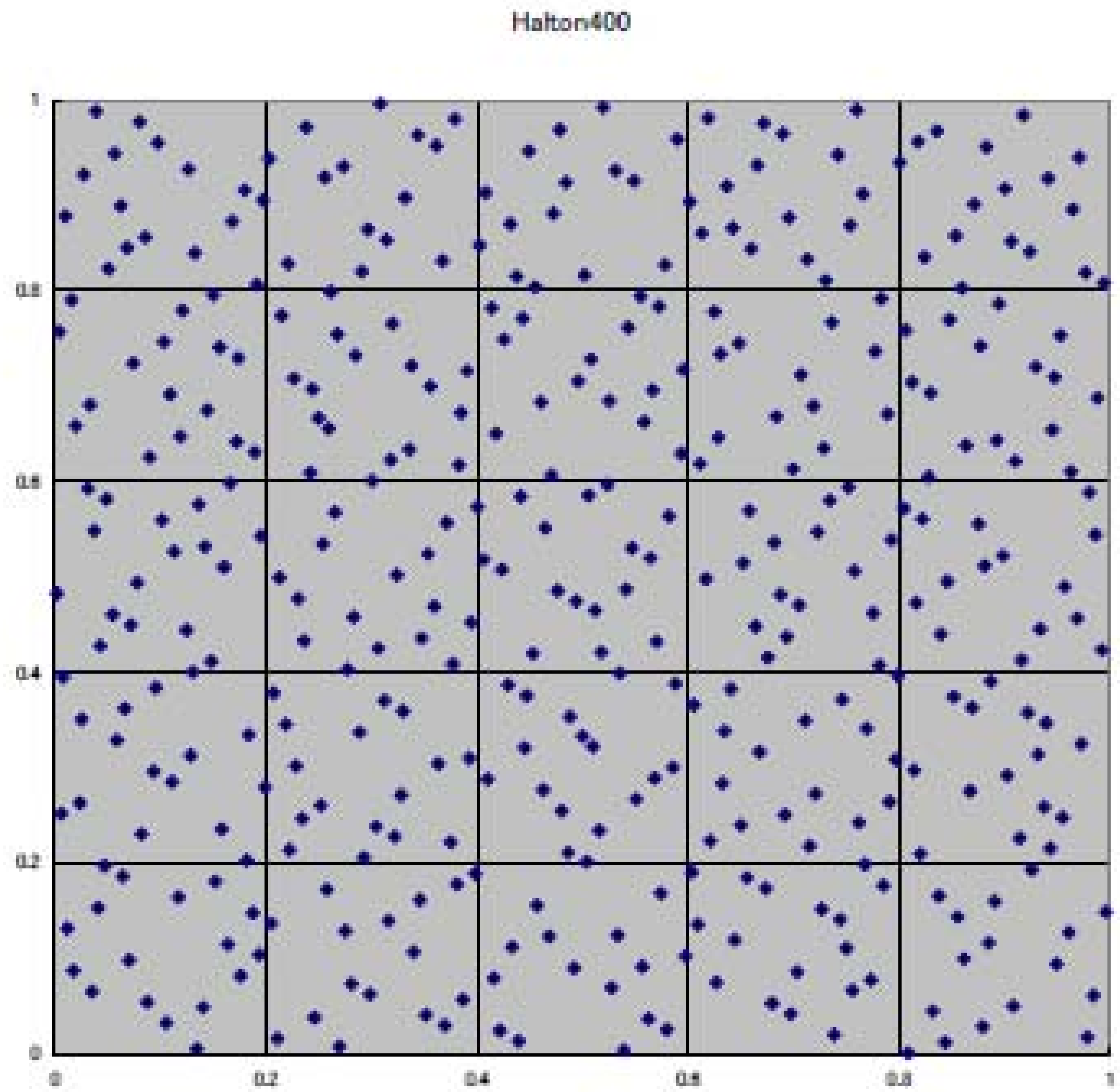
Then, there is a constant $C > 0$ that satisfies follows.

For any rectangle B whose sides are parallel to a square

$$D_N(B; \{(\phi_p(n), \phi_q(n))\}_{n=1}^{\infty}) \leqq C(\log N)^2, \quad N \geqq 1$$

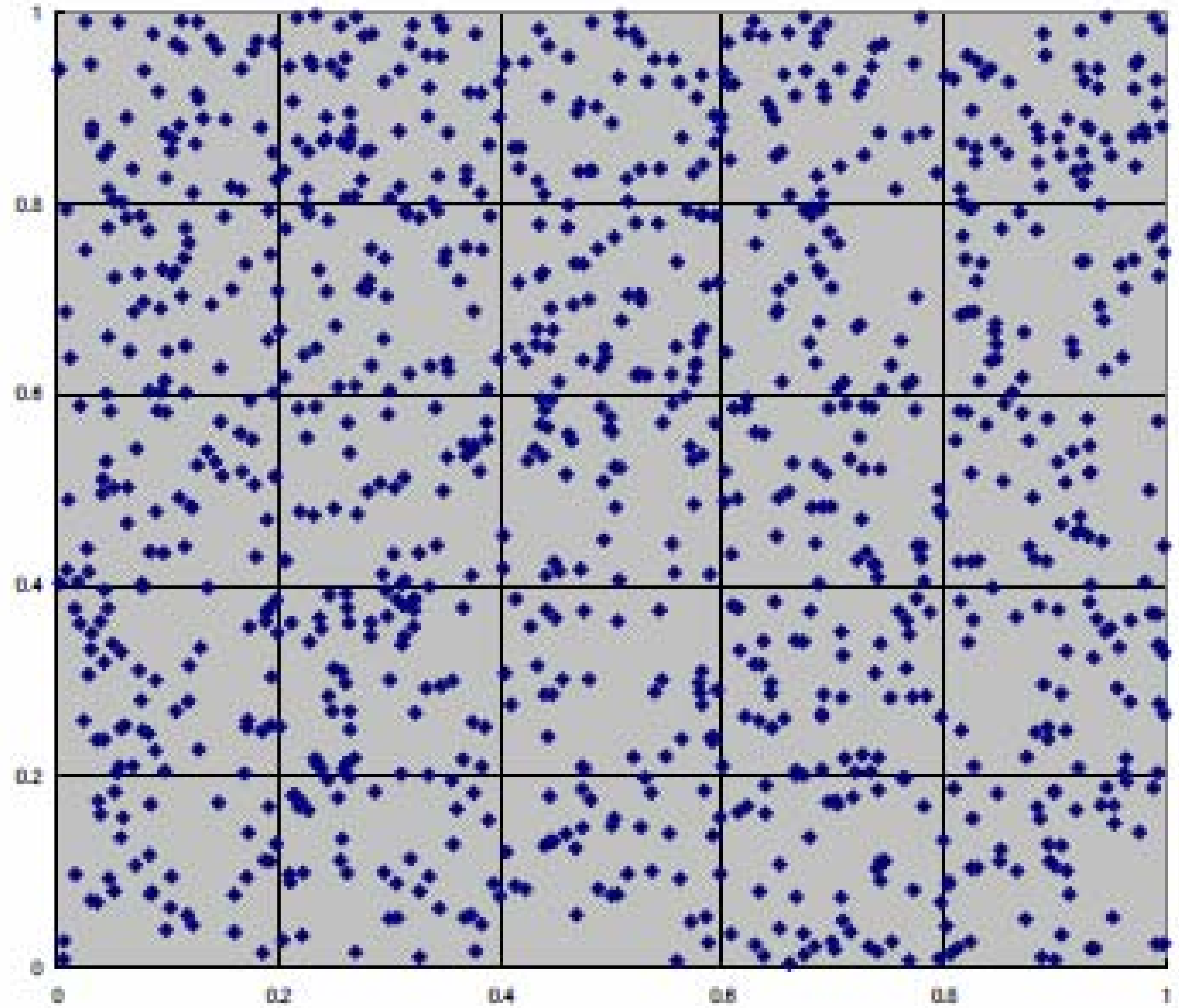
Random Number 400

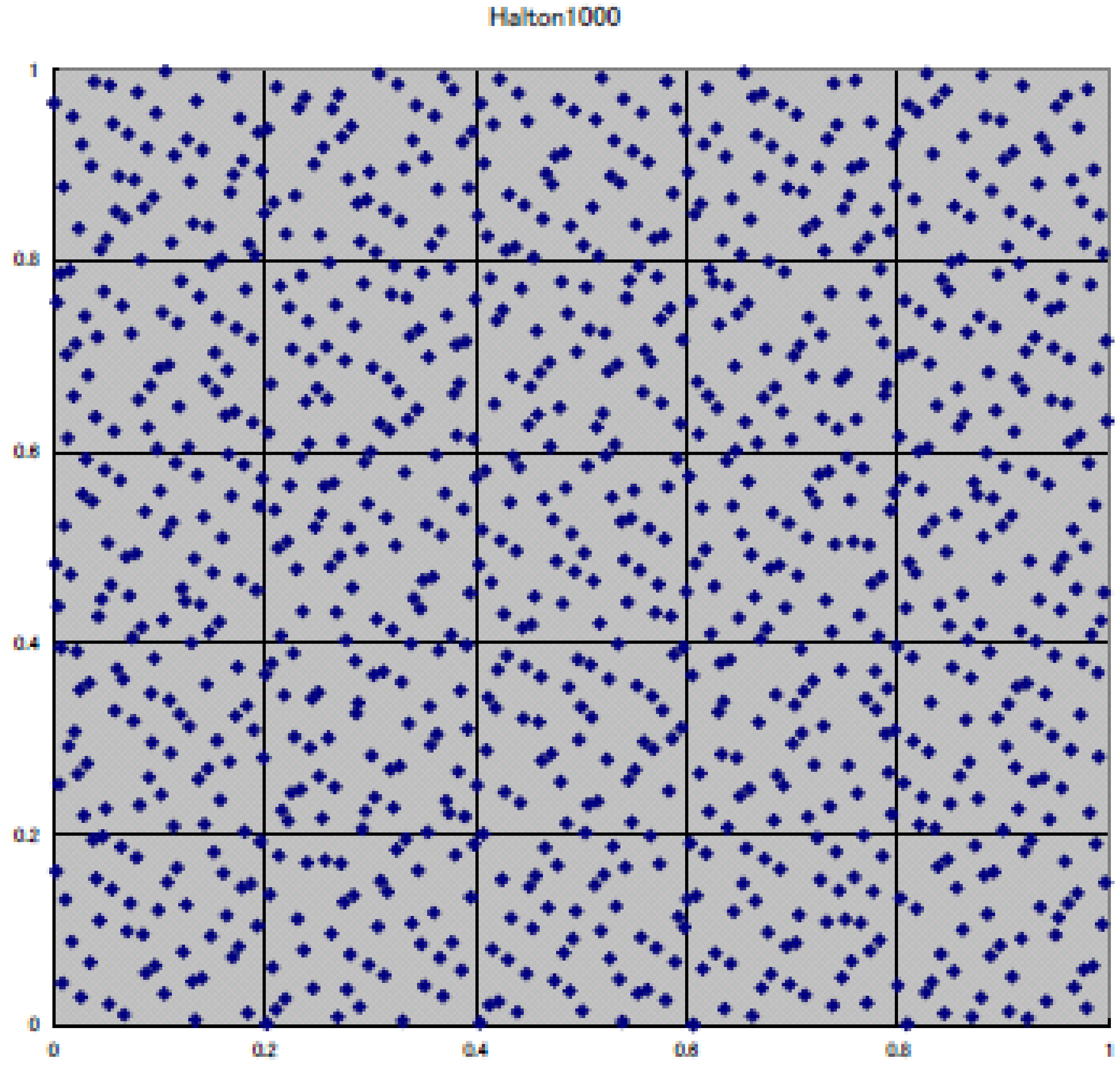




Halton400
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

Random Number 1000





Halton1000
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

For a vector sequence $\{(x_n, y_n, z_n)\}_{n=1}^{\infty}$ inside a cube $[0,1]^3$, a solid B in the cube, and natural number $N \geqq 1$,

$$D_N(B; \{(x_n, y_n, z_n)\}_{n=1}^{\infty})$$

$= |($ (x_n, y_n, z_n) in solid B, number of $1 \leqq n \leqq N$ $) - N \times$ (vol. of B)$|$

is determined. (Discrepancy)

Theorem 2 Suppose p, q, r are different prime numbers.

Then, there is a constant $C > 0$ that satisfies follows.

For any rectangle B whose sides are parallel to a square

$$D_N(B; \{(\phi_p(n), \phi_q(n), \phi_r(n))\}_{n=1}^{\infty}) \leqq C(\log N)^3, \quad N \geqq 1$$

When $d \geqq 1$

For a point sequence $\{z_n\}_{n=1}^{\infty}$, $B \in \mathcal{B}([0,1]^d)$ and $N \geqq 1$

$$D_N(B; \{z_n\}_{n=1}^{\infty})$$

$= |($ number of $n = 1, 2, \ldots, N$ satisfying $z_n \in B) - N \times (B$'s Lebesgue measure $)|$

is determined. (Discrepancy)

Theorem 3 Suppose $p_1, \ldots, p_d$ are different prime numbers.

Then, there is a constant $C > 0$ that satisfies follows.

For any $0 \leqq a_k \leqq 1, k = 1, \ldots, d,$

$$D_N([0, a_1) \times \cdots \times [0, a_d); \{(\phi_{p_1}(n), \cdots \phi_{p_d}(n))\}_{n=1}^{\infty}) \leqq C(\log N)^d, \quad N \geqq 1$$

A sequence that is evaluated as the theorem above is called **low discrepancy sequence**.

$\{z_n\}_{n=1}^{\infty}$ is a low discrepancy sequence.

$f : [0,1]^d \to \mathbf{R}$ a good function that satisfies a certain condition.

$$M_n = \frac{1}{n} \sum_{k=1}^{n} f(z_k)$$

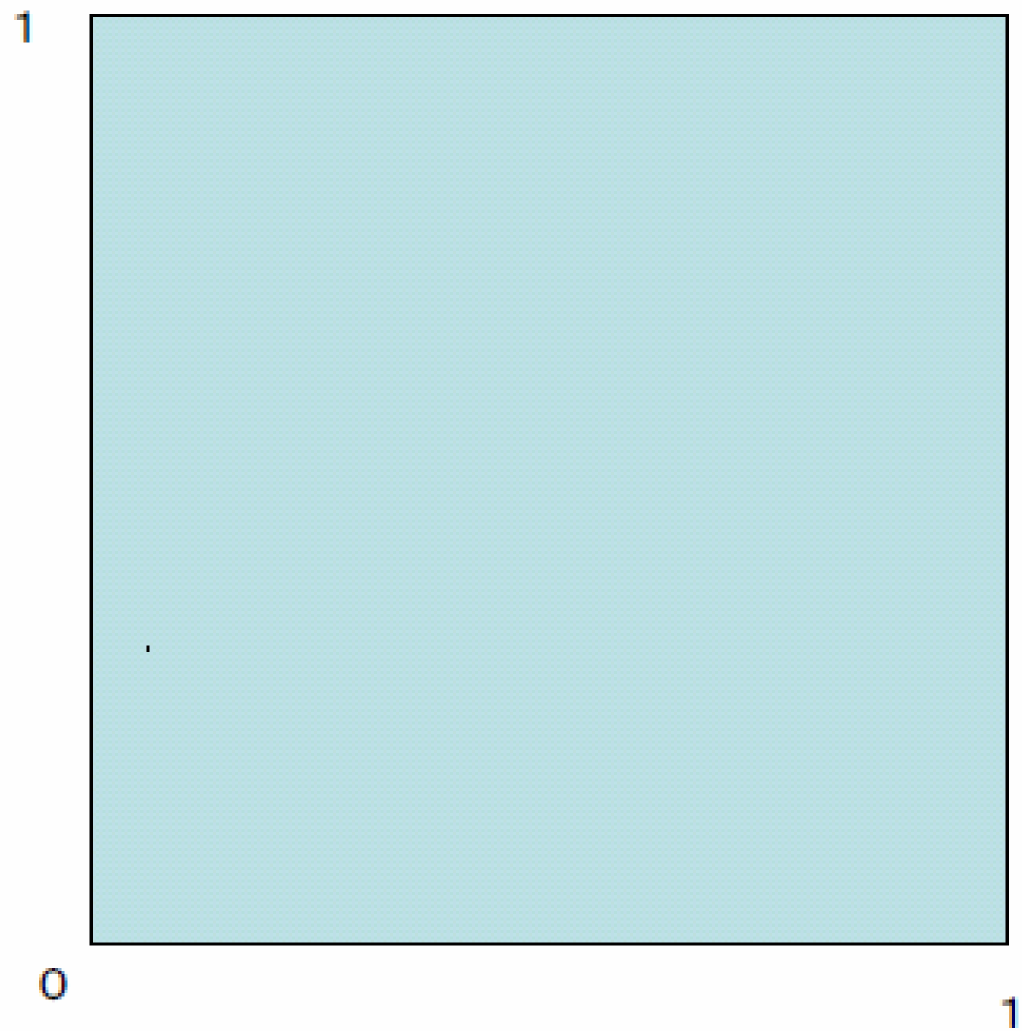
$$I = \int \cdots \int_{[0,1]^d} f(x) dx$$

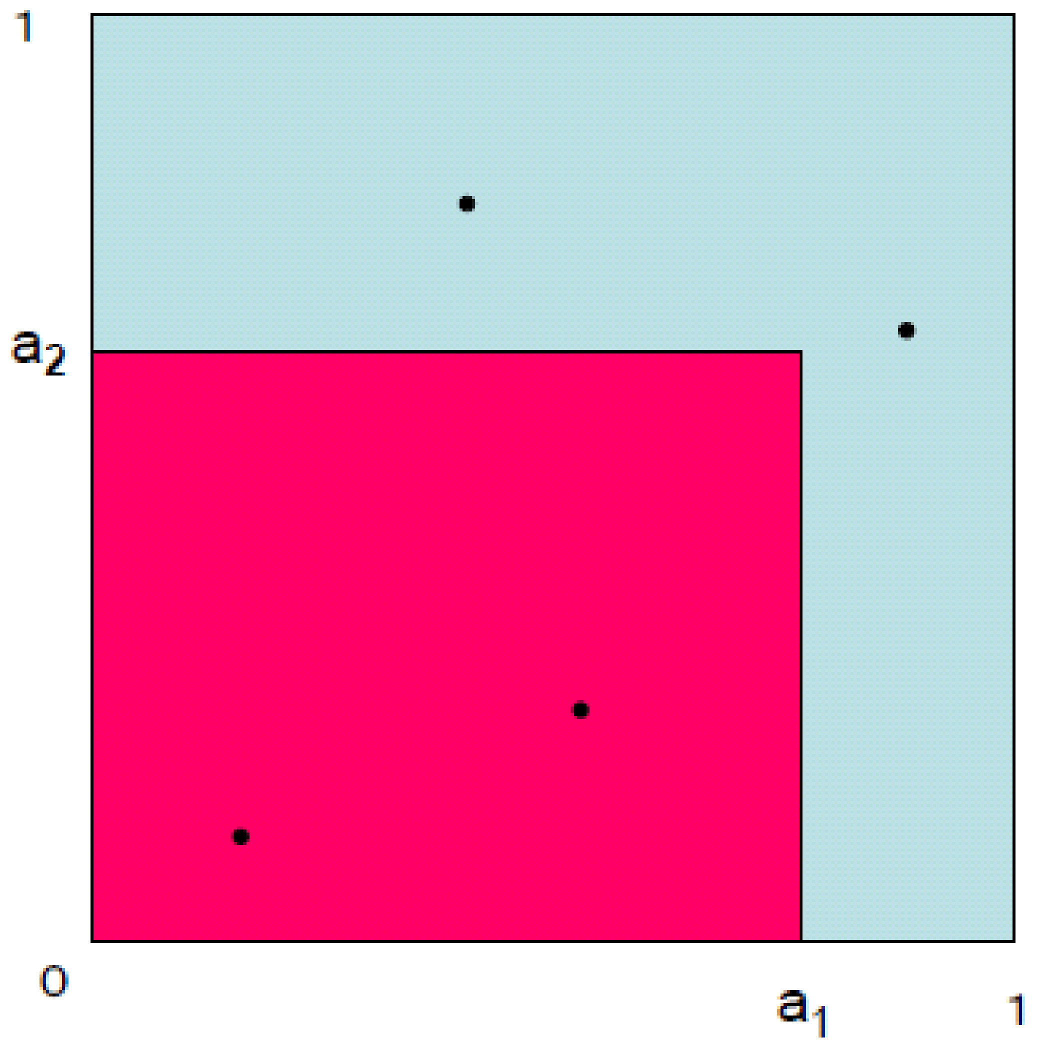
$$|I - M_n| \leqq C' \frac{(\log n)^d}{n}$$

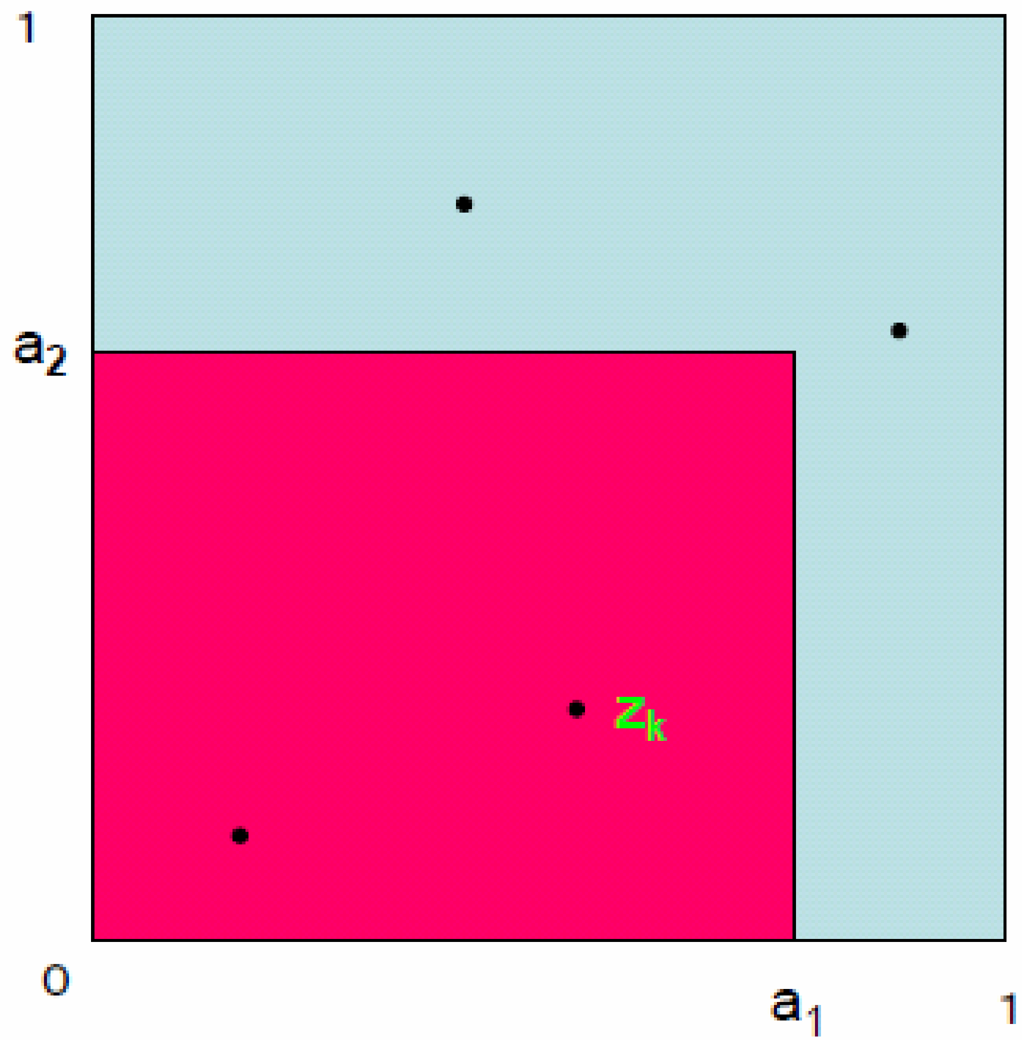
Error is almost in order of n^{-1} .

Halton sequence is logically OK, but actual efficiency is low in higher dimension.

Sobol et al. discovered the method using properties of polynomial expression of finite body successfully.

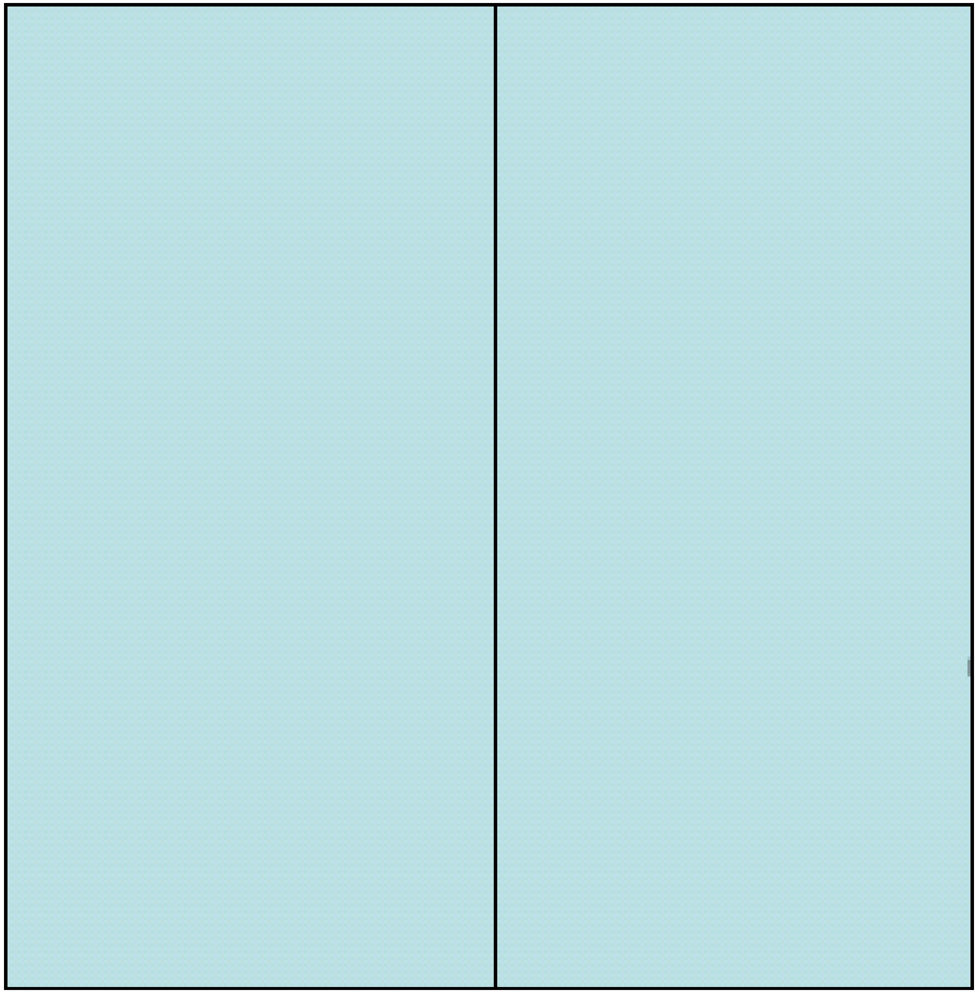

1
0
1


1
a2
0
a1
1

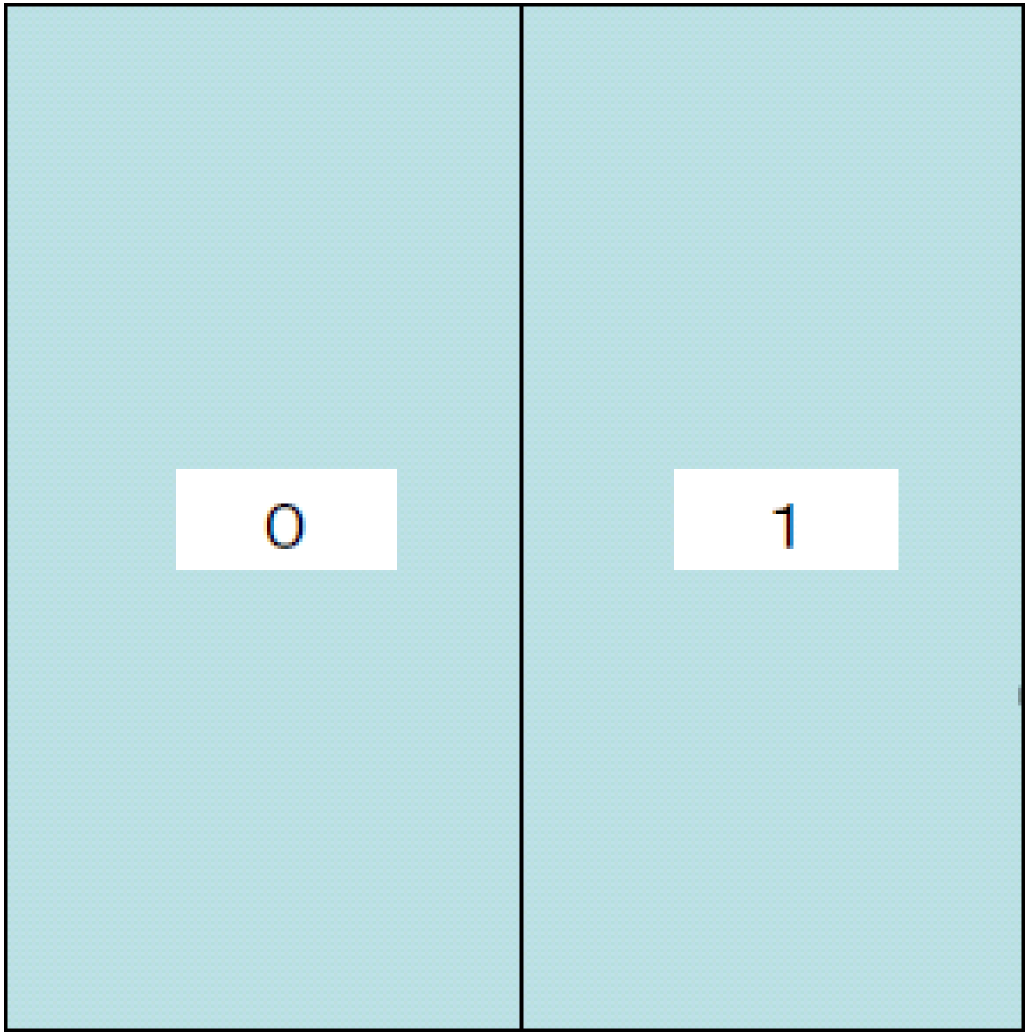


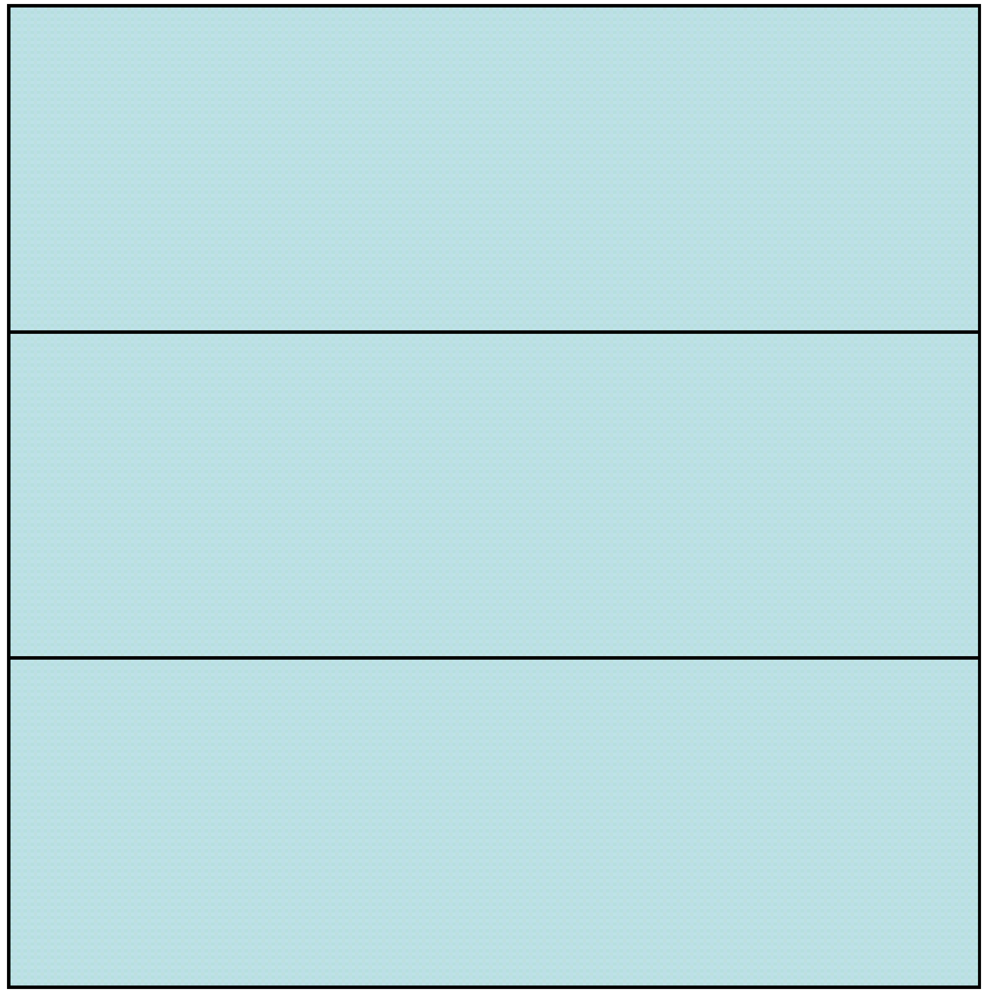
$k \leq N$

| [red box] -inside points' number - [red box] 's area × N |

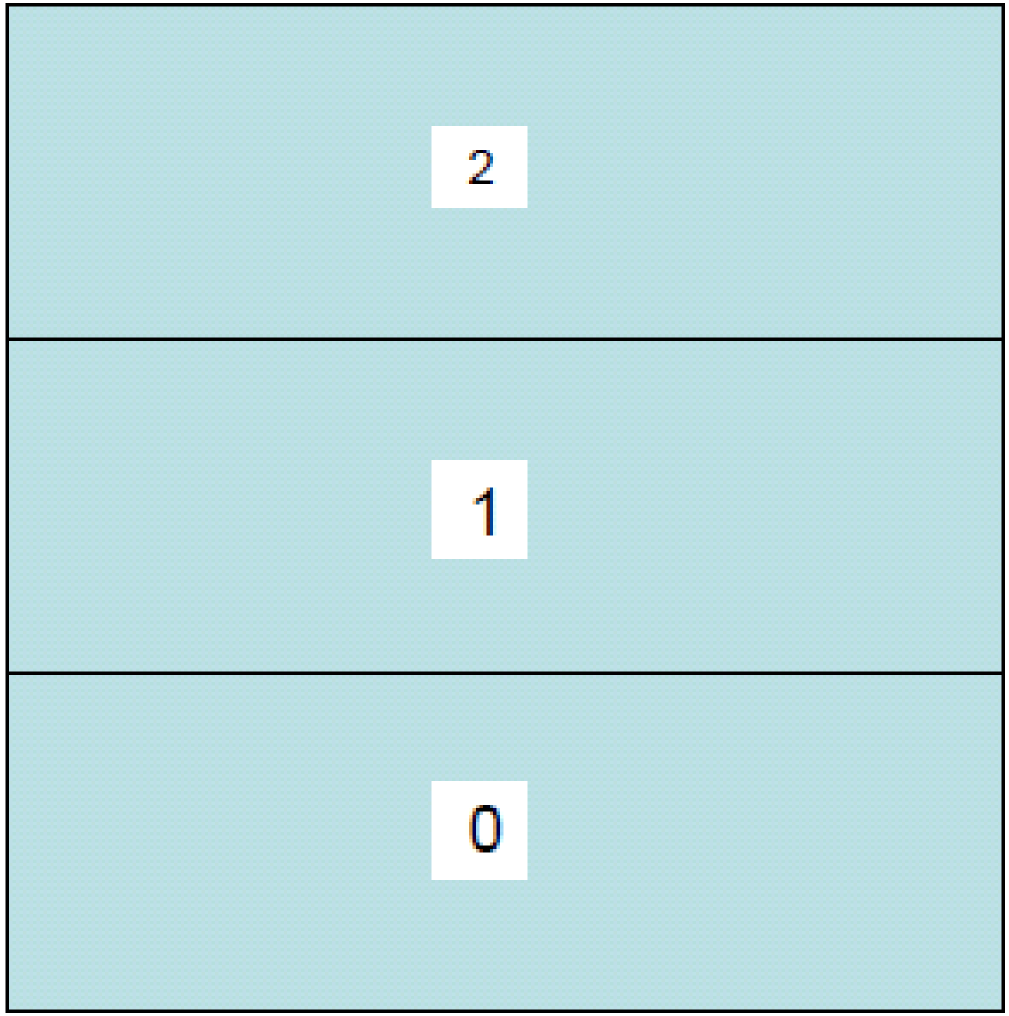


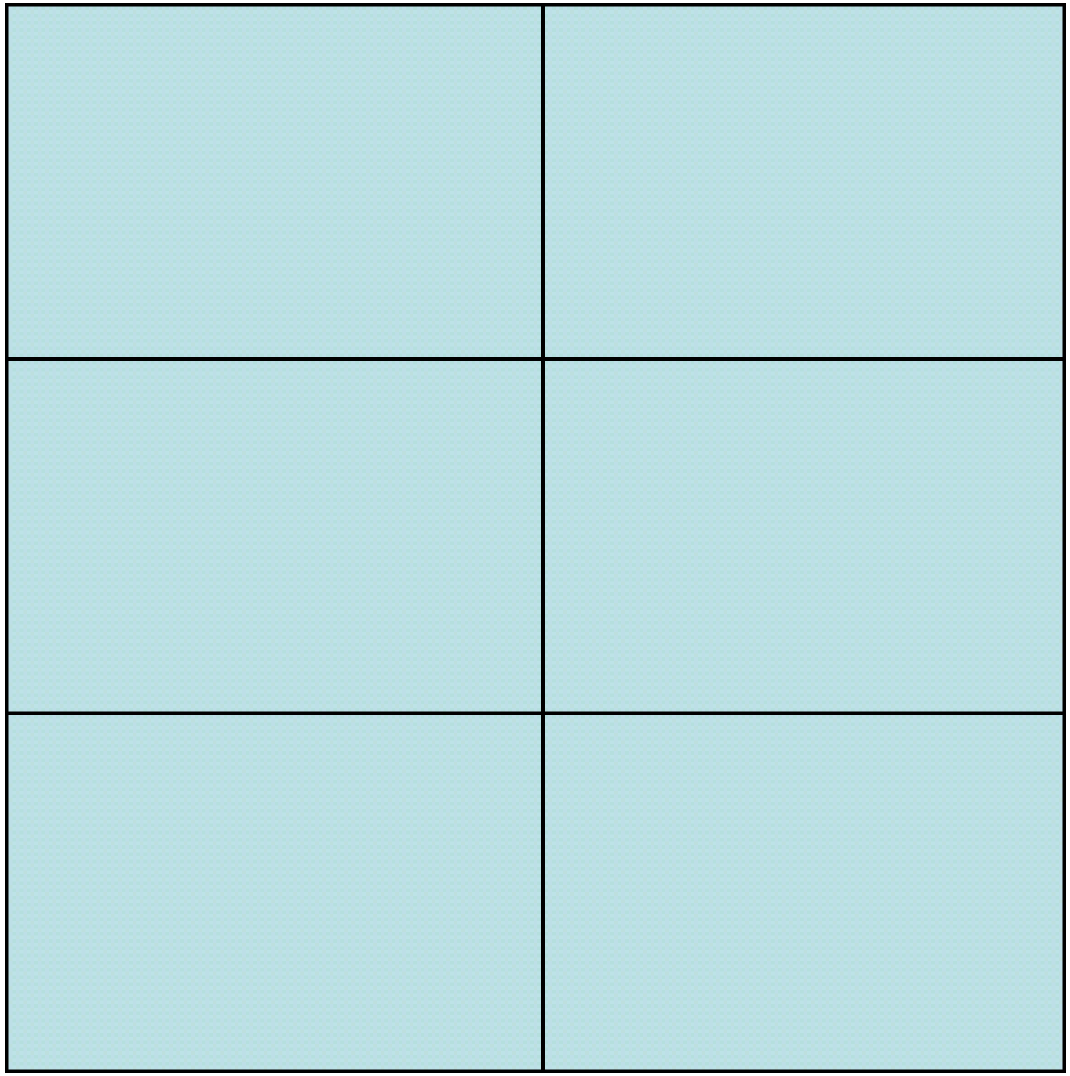
remainder
when n is
divided by 2



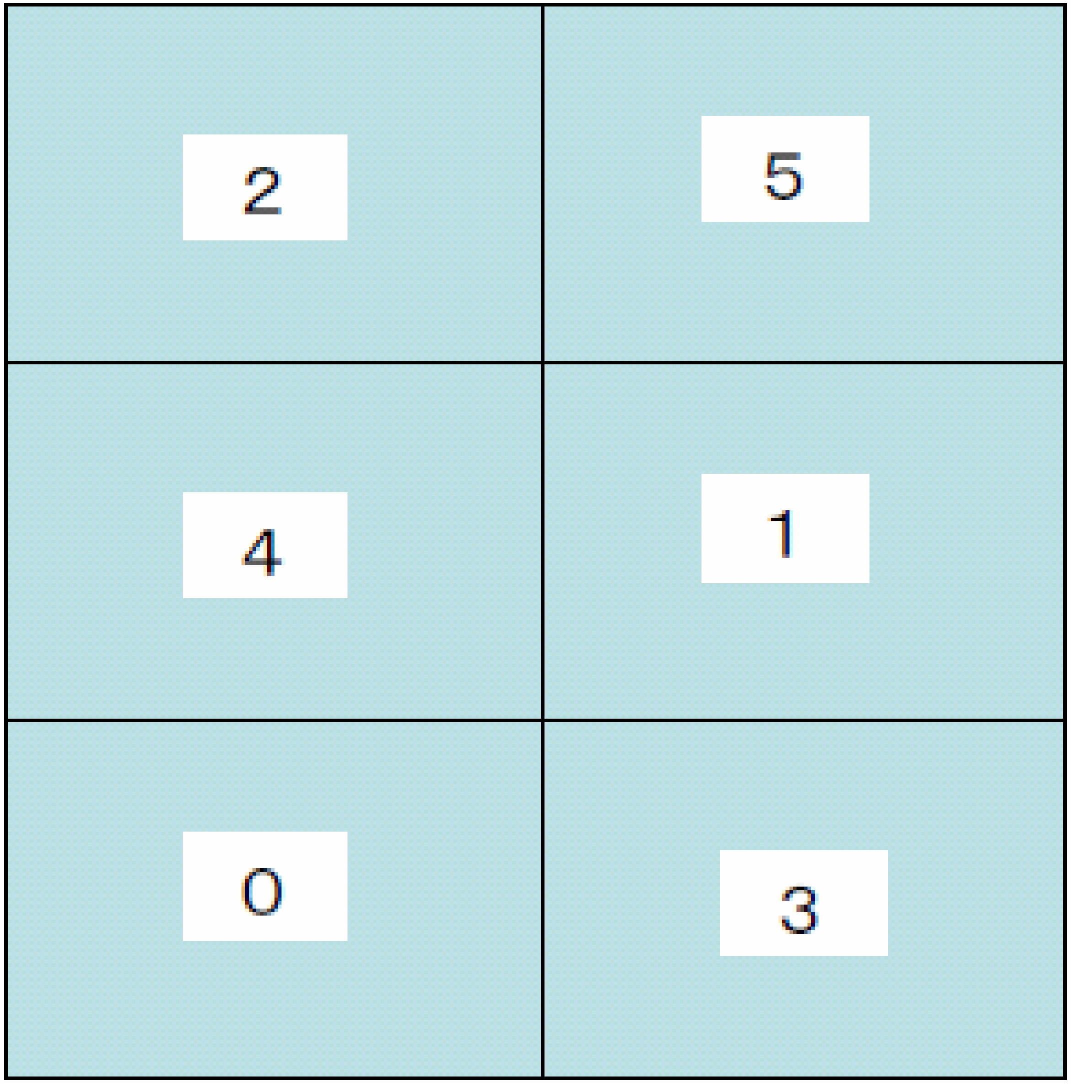


remainder
when n is
divided by 3

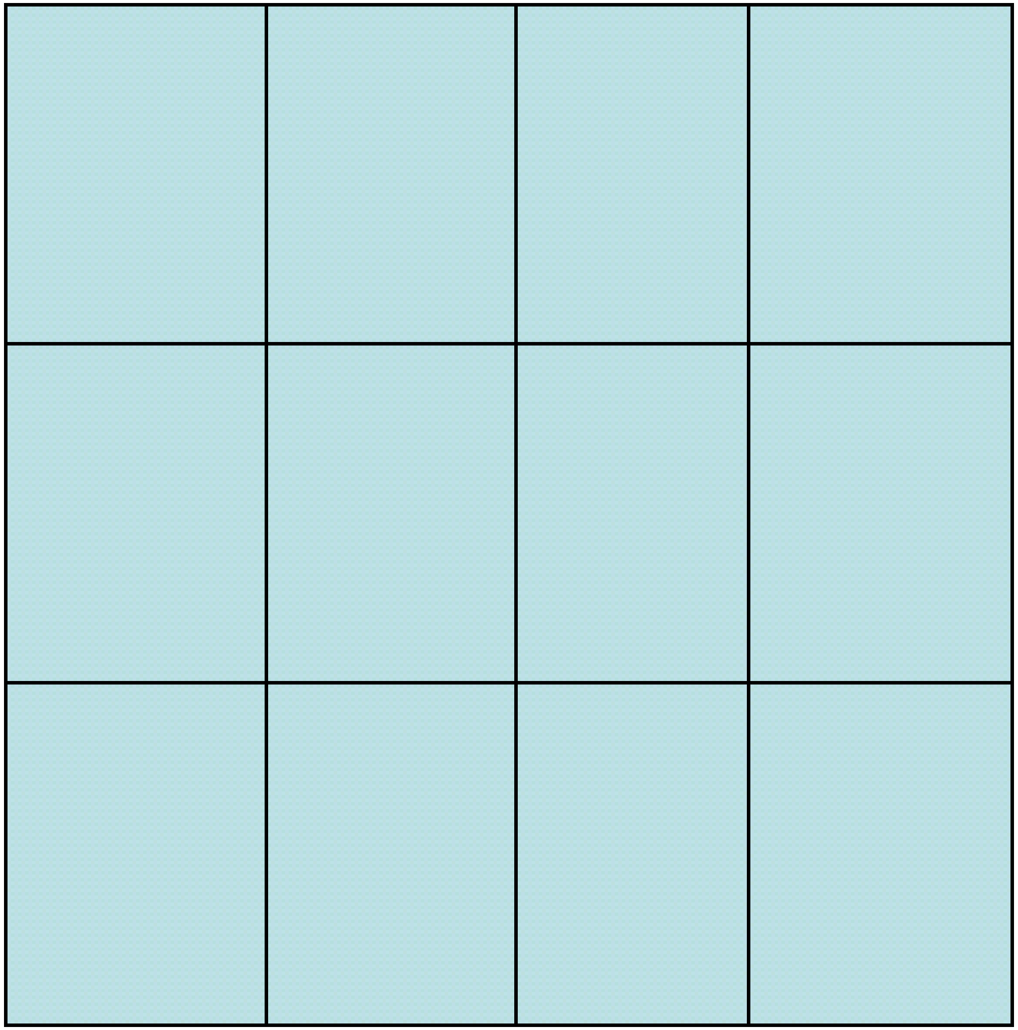




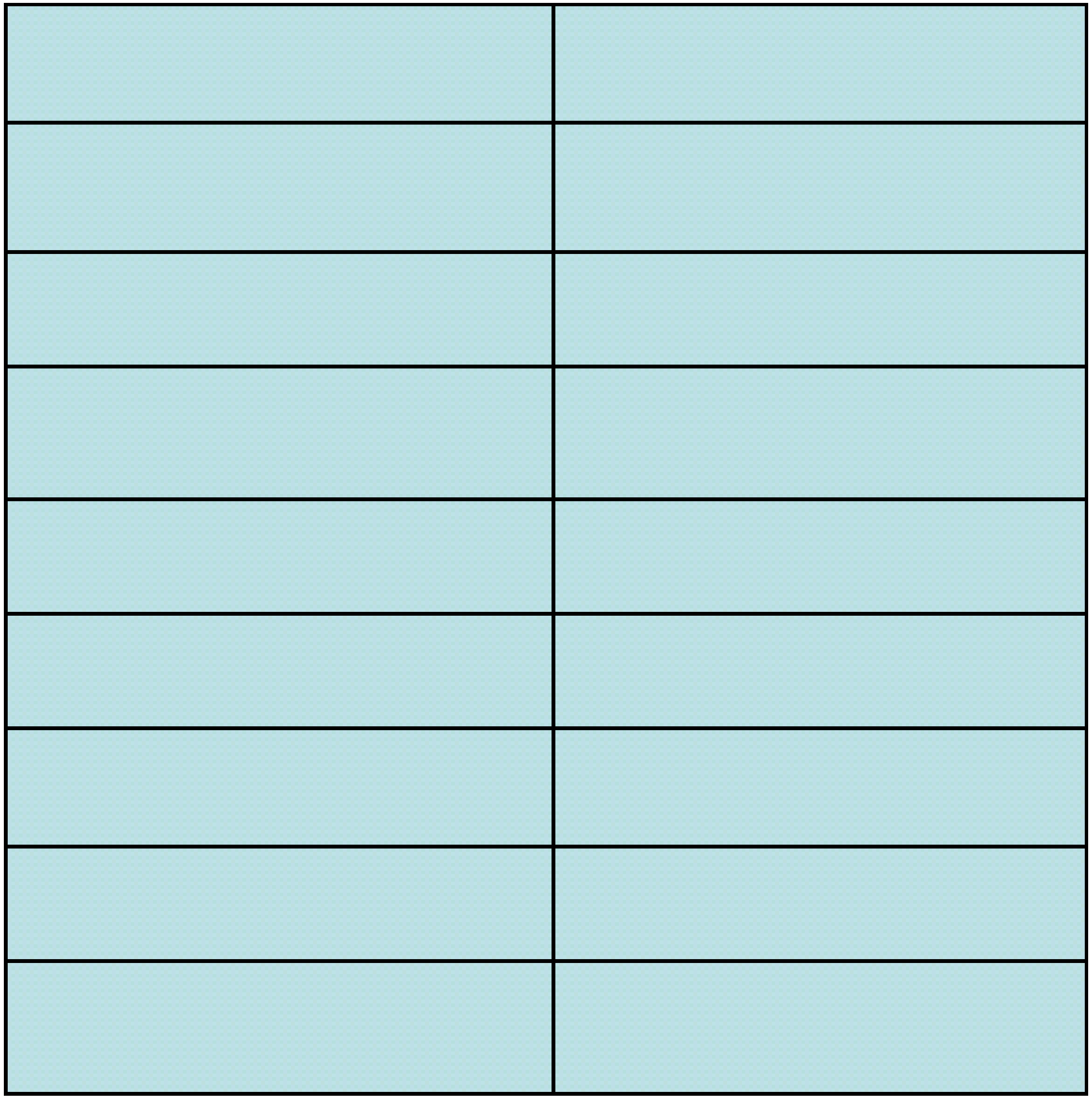
remainder
when n is
divided by 6



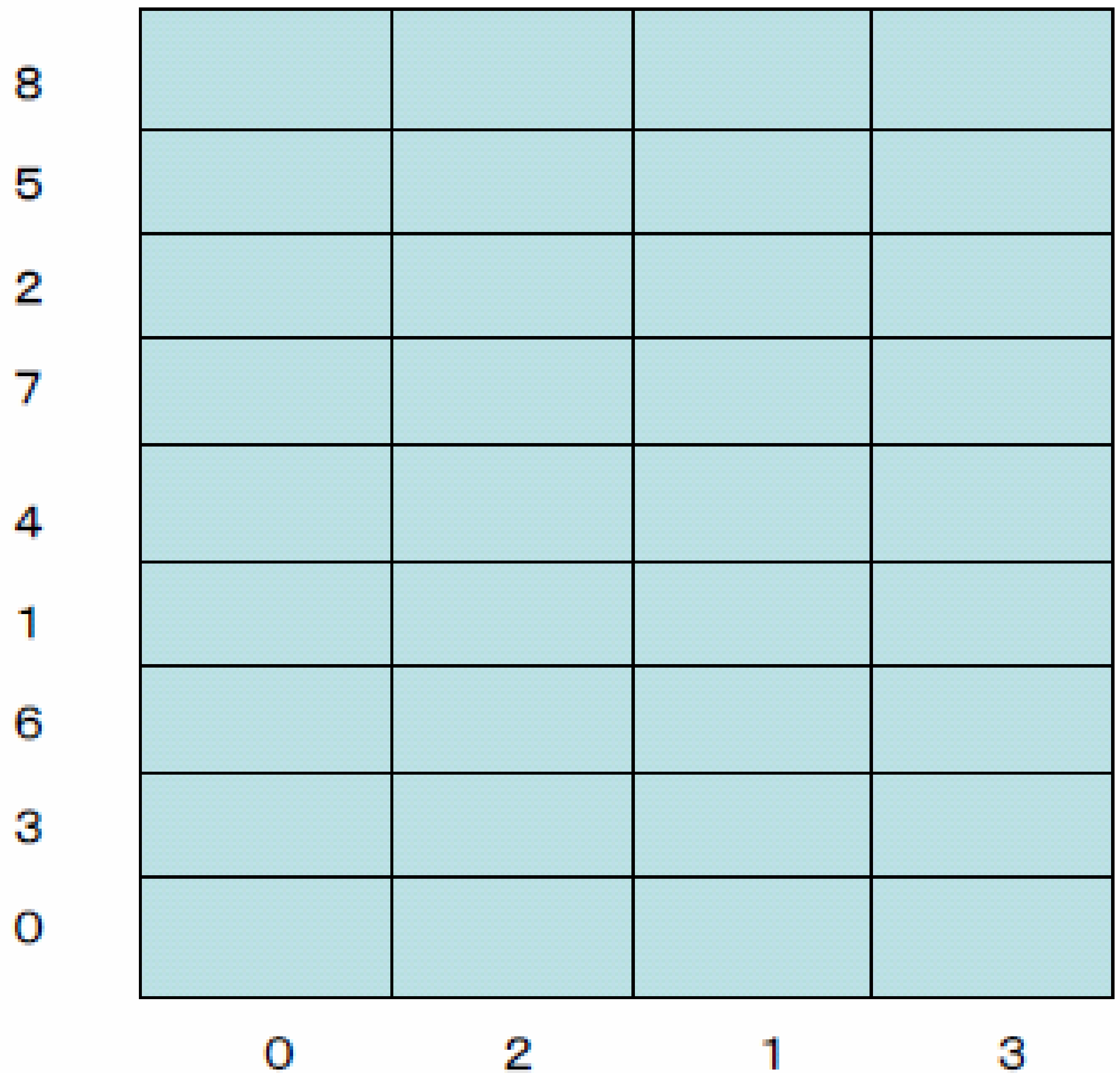
12 sections



18 sections



36 sections



36 sections

8				
5				
2				
7				
4				
1				
6				
3				
0				
	0	2	1	3