The World of Algebra
—Number Theory and Its Application—

#2 Digital Mathematics, Mathematics for Security
—Cryptographic theory—

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1. Cryptograph

cryptograph

Special code made to communicate secretly, which can be interpreted only by the parties involved.

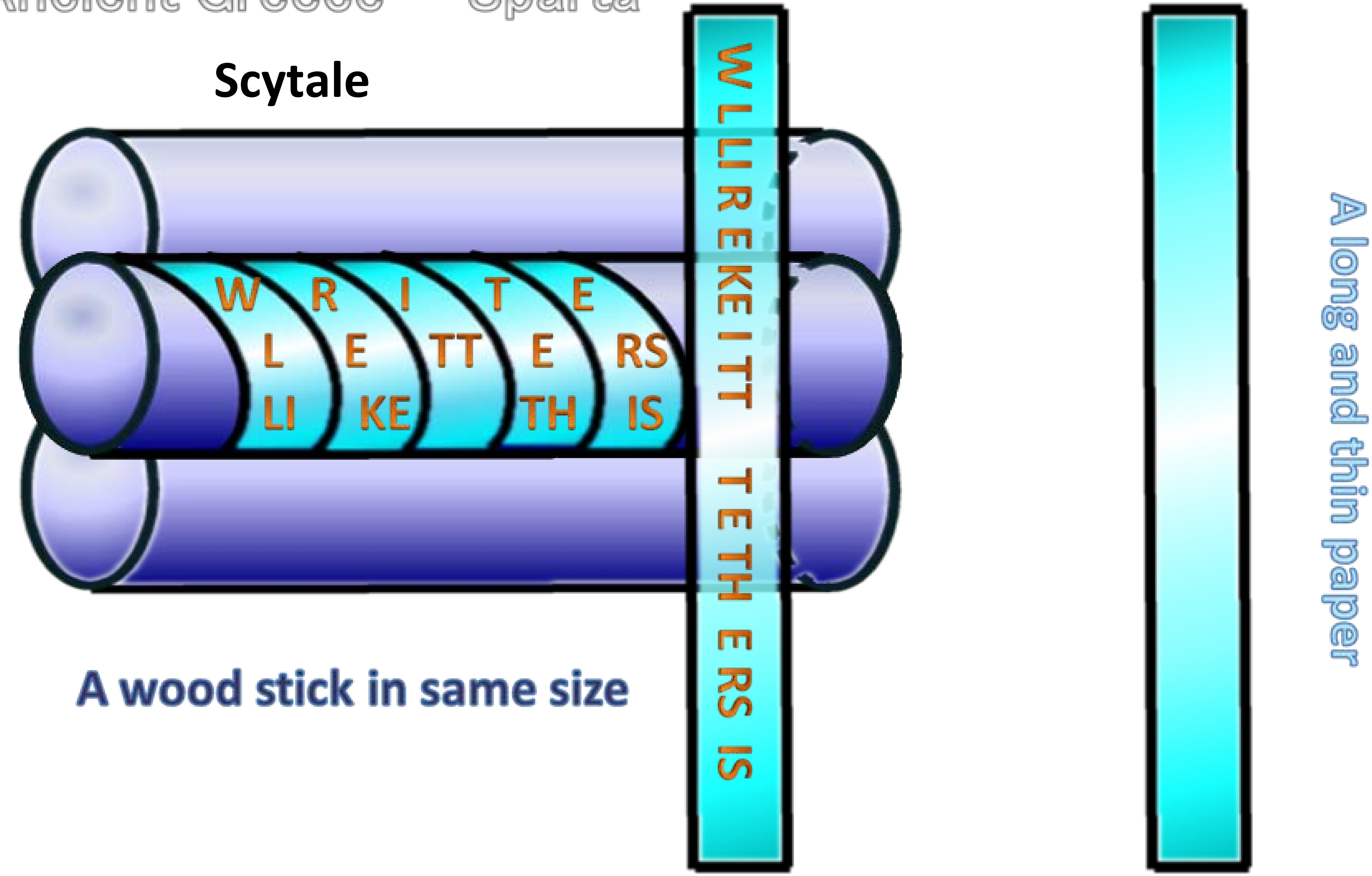
Or to make changes in sentences transmitted in a special way.

Here, the latter is dealt with.

Plain text	**Sentence to be transmitted**
Cryptograph	Changed sentence
Cipher	A **tool to decode cryptograph back into plain text**

Ancient Greece Sparta

Scytale



Twist a long and thin paper around a stick, and write letters on it along the stick

Caesar Cipher

Each letter is replaced by a letter some fixed number of positions further down the alphabet.

Ex. shift of 1

plain	HUKANKOUGI
cryptograph	IVLBOLPVHJ
cipher	shift of 1

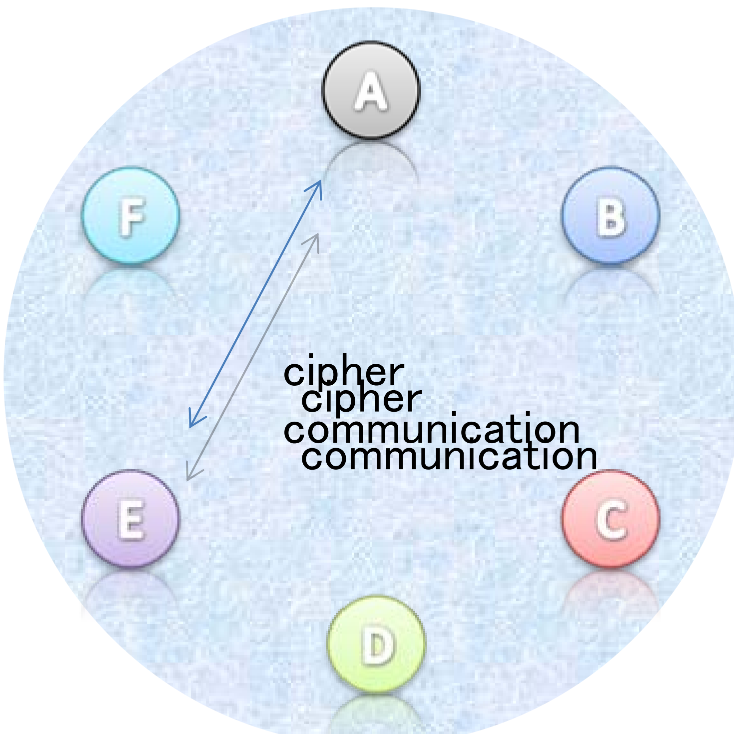
Symmetric-Key Cryptography

A cryptography whose cipher is shared by sender and receiver.

1976 W. Diffie, M.Hellman

It was discovered that cipher communication between two unspecified people is possible even when a cipher is open to general public.

Public-key cryptography



Key is widely distributed.

The Principle of Public-Key Cryptography

Even the quickest computer takes an inordinate amount of time to solve its algorithm, decoding is practically impossible.

Organic law

(1) By a difficult prime-factor-decomposed computation

(2) By a difficult discrete logarithm problem

RSA cryptography using (1)is to be introduced.

2. Theorems of Number Theory

theorem (Fermat's little theorem)

if p is a prime number and a is any integer that does not have p as a factor,

$$a^{p-1} \equiv 1 (\mathrm{mod}\ p)$$

proof

Let us assume that $\{\overline{1}, \overline{2}, \cdots, \overline{p-1}\}$ **is a member of** $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ **not including** $\overline{0}$

Since a **is not divisible by** p **,**

$$\{\overline{1}, \overline{2}, \cdots, \overline{p-1}\} = \{\overline{a} \cdot \overline{1}, \overline{a} \cdot \overline{2}, \cdots, \overline{a} \cdot \overline{p-1}\}$$

therefore,

$$\begin{aligned} \overline{1} \cdot \overline{2} \cdot \ \cdots \ \cdot \overline{p-1} &= \overline{a}\,\overline{1} \cdot \overline{a}\,\overline{2} \cdot \ \cdots \ \cdot \overline{a}\,\overline{p-1} \\ &= \overline{a}^{p-1} \cdot \overline{1} \cdot \overline{2} \cdot \ \cdots \ \cdot \overline{p-1} \end{aligned}$$

$\overline{1} \cdot \overline{2} \cdot \ \cdots \ \cdot \overline{p-1}$ **is** $\mathbb{F}_p$ **and not** $\overline{0}$

$$\overline{a}^{p-1} = \overline{1}$$

Ex. 1

$p = 71$ **prime**

$a = 1687$ **is not divisible by** p.

$$1687^{70} \equiv 1 \pmod{71}$$

Ex. 2

$q = 97$ **prime**

$a = 1687$ **is not divisible by** q.

$$1687^{96} \equiv 1 \pmod{97}$$

Generalize Fermat's Little Theorem Slightly, and Use it in a Code.

p, q 2 prime numbers

$\mathbb{Z}/pq\mathbb{Z}$ commutative ring

theorem

If a **is an integer coprime to** pq **, then**

$$a^{(p-1)(q-1)} \equiv 1 (\text{mod } pq)$$

example

Suppose $p = 71, q = 97, pq = 6887$

Then

$$(p-1)(q-1) = 6720$$

$a = 1687$ **cannot be divided by** p, q

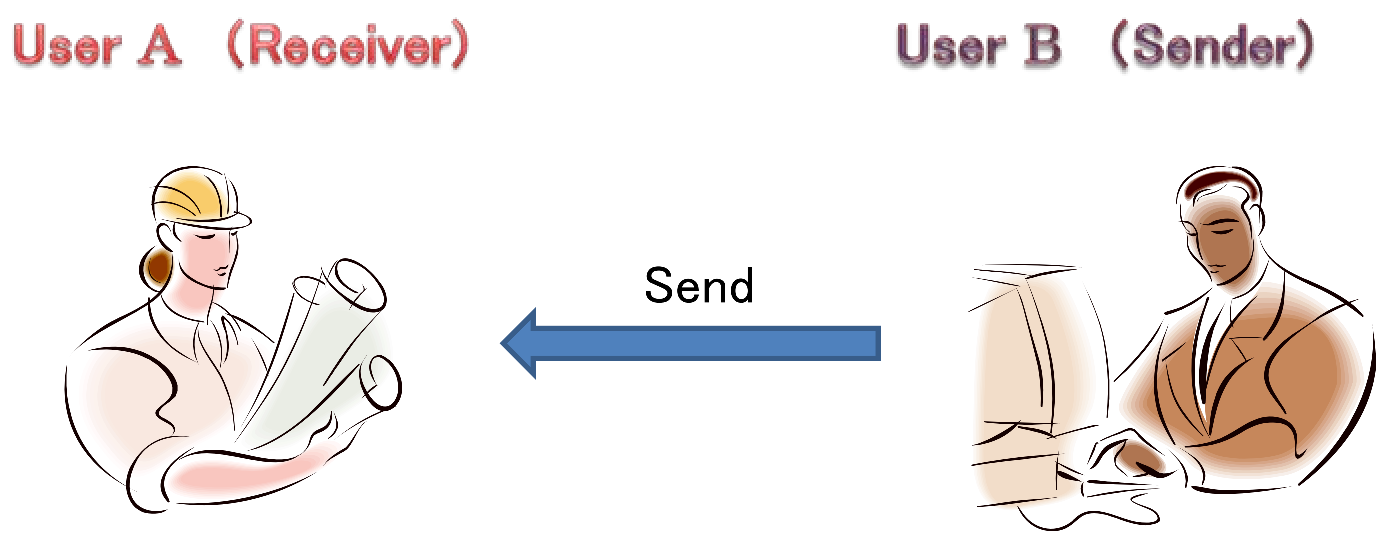
$$1687^{6720} \equiv 1 \ (\text{mod } 6887)$$

3. RSA public key encryption

1978

Presented by R.Rivest, A.Shamir, L.N.Adleman

- Public key cryptography using the fact that it is difficult to factorize product of 2 large prime numbers.



User A（Receiver）

Selects 2 large primes, p, q

and compute $n = pq$

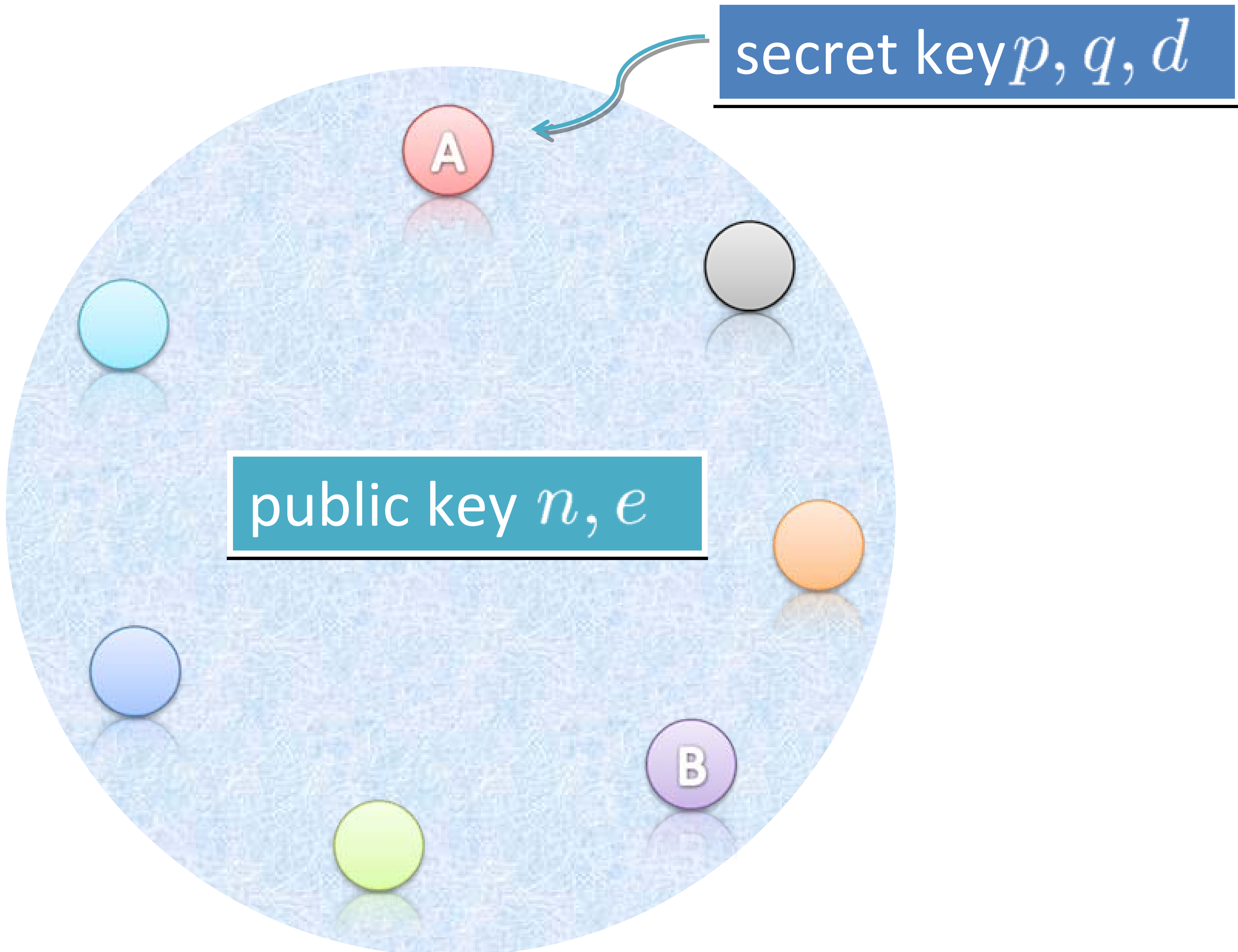
Chooses an integer $\overline{e}$ from $\mathbb{Z}/(p-1)(q-1)\mathbb{Z}$

such that an integer d exists when

$$ed \equiv 1 \pmod{(p-1)(q-1)}$$

public key n, e

secret key p, q, d



secret key p, q, d
A
public key n, e
B

User B (Sender)

Send a plain sentence M which is coprime to n .

encryption M^e

principl
e

Third person cannot calculate d since prime factorization of n is extremely difficult.

User A (Receiver)

reconstruct M from $(M^e)^d \equiv M \pmod{n}$

Verification

From generalization of Fermat's little theorem,

$$M^{(p-1)(q-1)} \equiv 1 \pmod{n}$$

Since $ed \equiv 1 \pmod{(p-1)(q-1)}$**, there is an integer** s

$$ed = (p-1)(q-1)s + 1$$

Therefore,

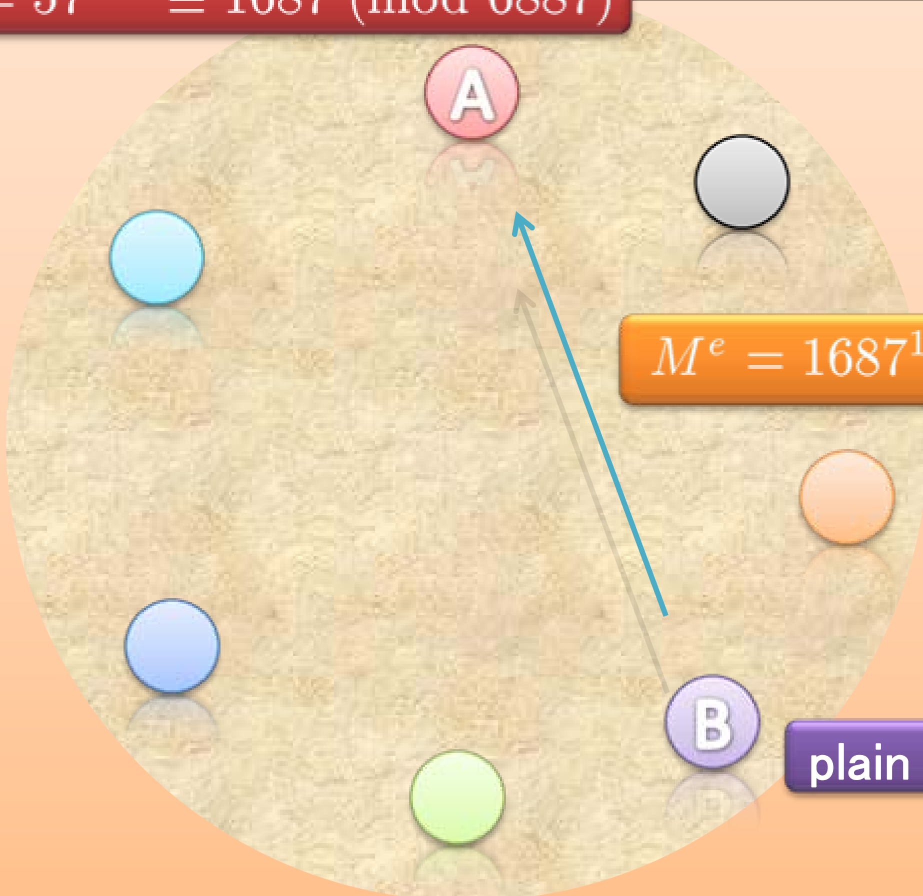
$$\begin{aligned} M^{ed} \pmod{n} &\equiv M^{(p-1)(q-1)s+1} \pmod{n} \\ &\equiv (M^{(p-1)(q-1)})^{s} M \pmod{n} \\ &\equiv M \pmod{n} \end{aligned}$$

Example

secret key: $p = 71, q = 97, d = 517$

public key : $n = pq = 6887, e = 13$

$57^d = 57^{517} \equiv 1687 \pmod{6887}$



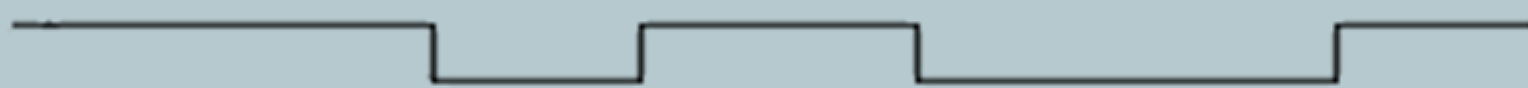
$M^e = 1687^{13} \equiv 57 \pmod{6887}$

plain sentence $M = 1687$

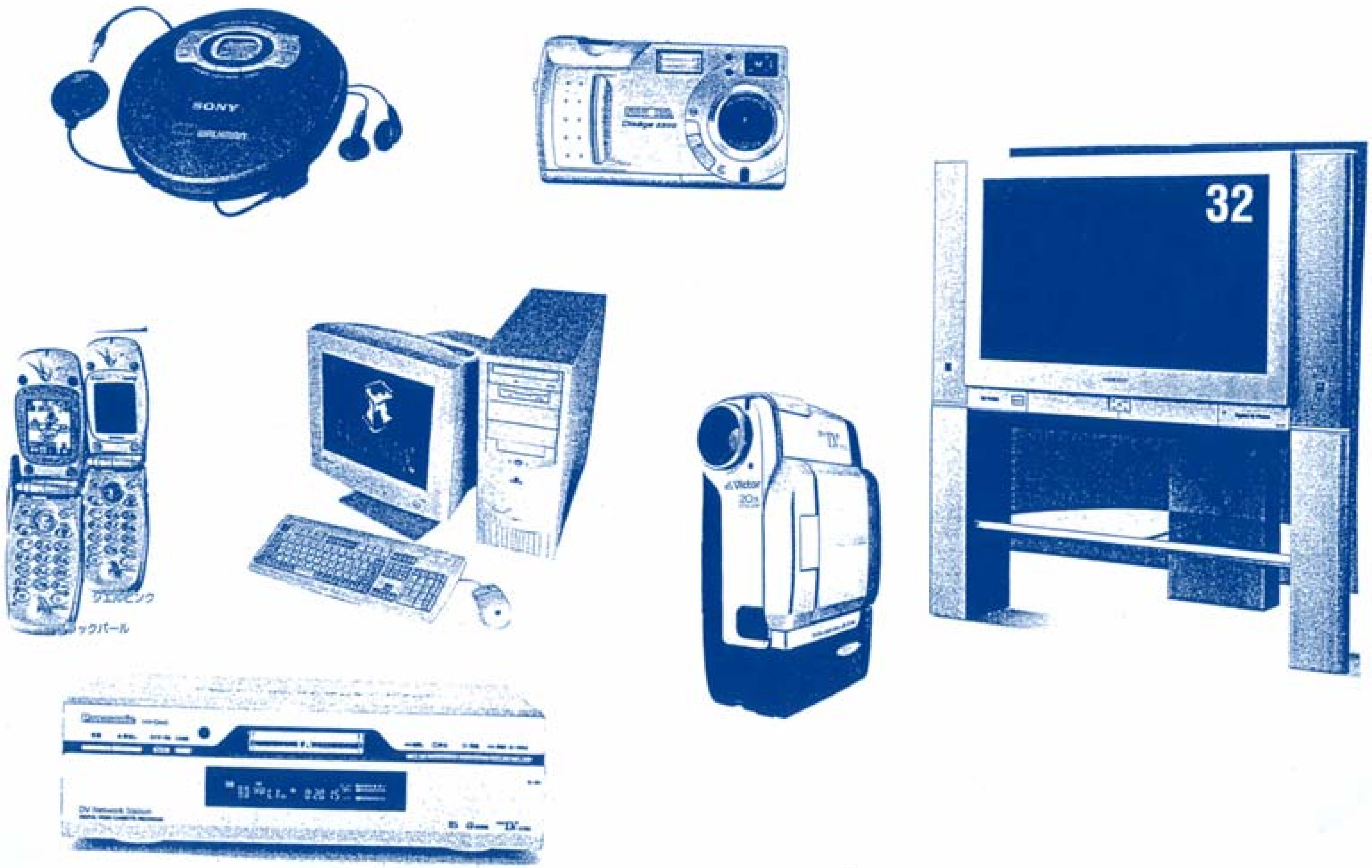
4. Code Theory

Analog to Digital

Compact Disc



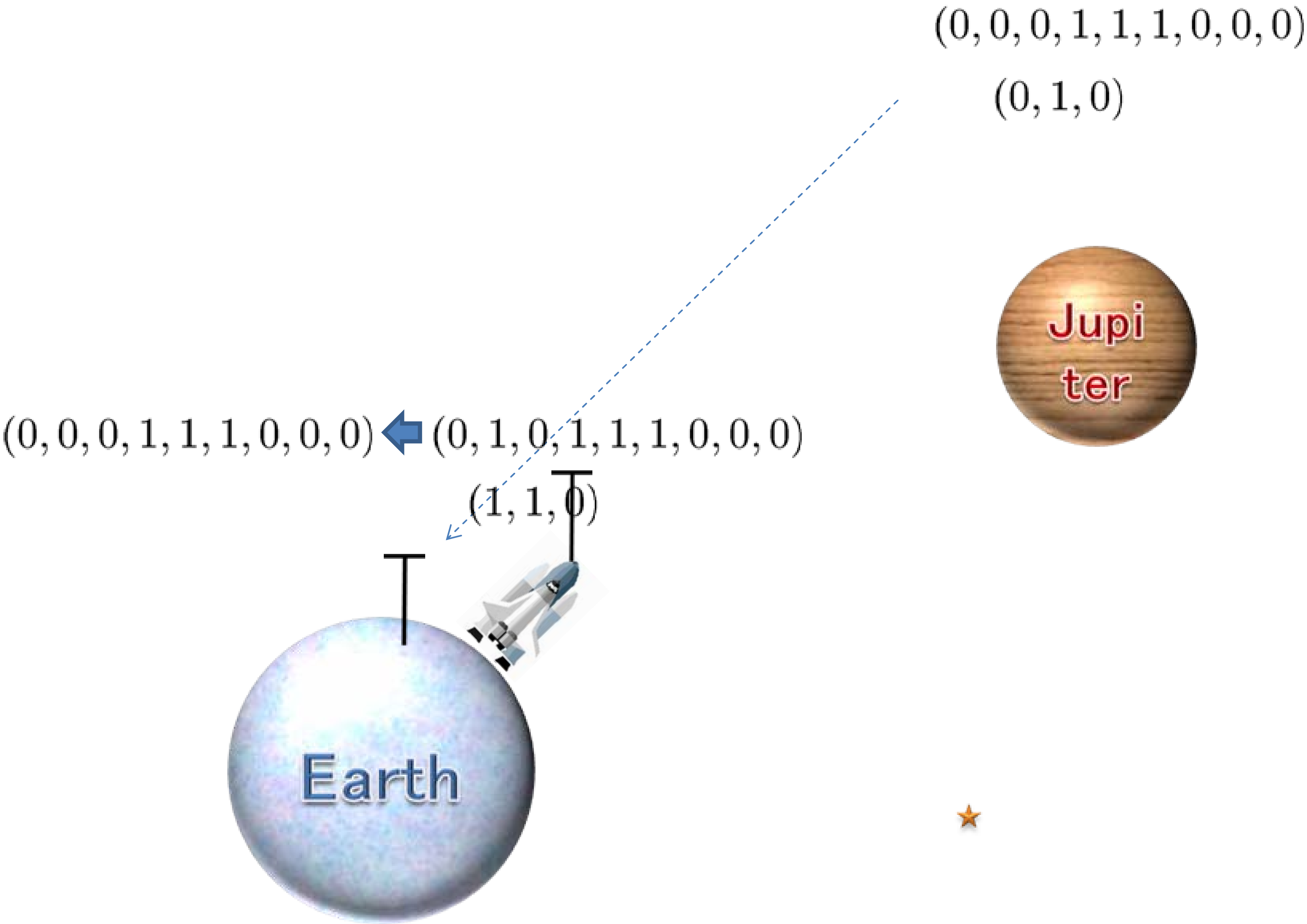
0 0 0 0 0 1 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0


SONY
ジェルピンク
Victor
32
DV Network Station

Where Digital is, Error Exists

Error-correction code

Error-detection code



$(0,0,0,1,1,1,0,0,0)$
$(0,1,0)$
Jupi
ter
$(0,0,0,1,1,1,0,0,0)$
$(0,1,0,1,1,1,0,0,0)$
$(1,1,0)$
Earth


Information source
coder
Transmission channel
decoder
destination

$\mathbb{F}_q^n$: n -dimensional vector space in $\mathbb{F}_q$.
$(x_1, x_2, \cdots, x_n)$ are called **alphabets**.

$\mathbb{F}_q^n \supset C$
C is called **code**.
Members of C are used as alphabets.
n : word length of C .

$\mathbb{F}_q^n \setminus C$: redundancy
This is used for error-correction.

The larger C is, the more information can be transmitted.

Ability of error-correction is generally higher if $\mathbb{F}_q^n \setminus C$ is larger.

More efficient code that fulfills both of these conflicting conditions is desired.

Distance of $\mathbb{F}_q^n$ is Defined to Check Errors.

Distance

A distance between 2 points $P = (x_1, x_2, x_3)$ **and** $Q = (y_1, y_2, y_3)$ **in 3-dimensional Euclidean space** $\mathbb{R}^3$

$$d(P, Q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

that fulfills 3 conditions below.

(i) $d(x, y) \geq 0.$ or $d(x, y) = 0 \Leftrightarrow x = y.$

(ii) $d(x, y) = d(y, x)$

(iii) [triangle inequaity] $d(x, y) + d(y, z) \geq d(x, z)$

These 3 conditions are essential for distance.

In other words, if these 3 conditions are fulfilled, it can be called a distance.

Definition

distance of $\mathbb{F}_q^n \ni x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$

(Hamming distance)

$$d(x, y) = \sharp\{1 \leq i \leq n \mid x_i \neq y_i\}$$

Here, $\sharp S$ **is a number of members in group** S **.**

That this fulfills the axiom of distance space can be easily verified.

For example, in $\mathbb{F}_2^5$ **, 4 components are different in** $x = (1, 0, 1, 0, 1)$ **and** $y = (1, 1, 0, 1, 0)$ **, so** $d(x, y) = 4$ **.**

In other words, if $x = (1, 0, 1, 0, 1)$ **is sent and** $y = (1, 1, 0, 1, 0)$ **is received, error with Hamming distance of** $d(x, y) = 4$ **has occurred.**

Definition

When $z \in \mathbb{F}_q^n$, and r is a natural number.

$$B_r(z) = \{x \in \mathbb{F}_q^n \mid d(z,x) \leq r\}$$

is a sphere with a radius r from the center z .

definition

minimal distance d

When C is a subset of $\mathbb{F}_q^n$, its minimal distance d is defined as :

$$d = \min\{d(x,y) \mid x, y \in C,\ x \neq y\}$$

Here, $\min$ stands for minimum value.

Ex.

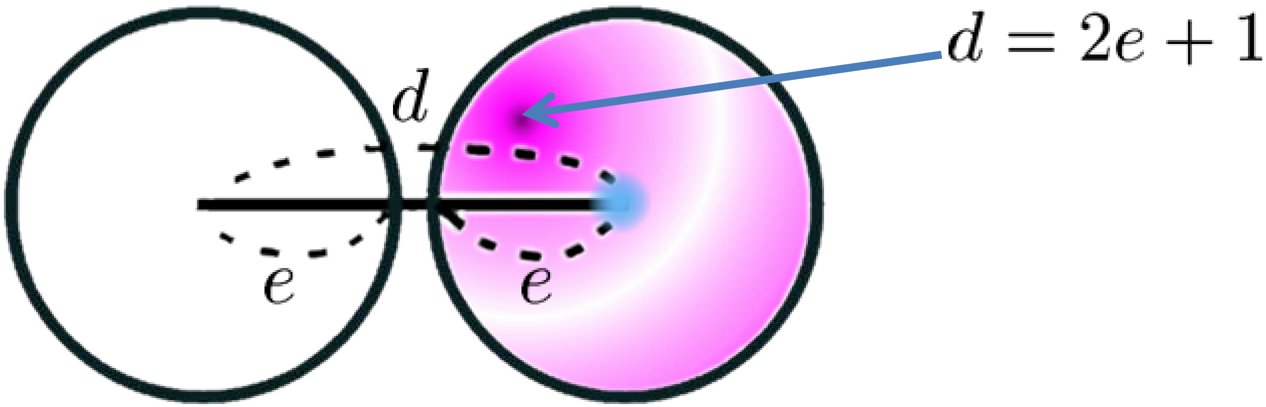
When $\mathbb{F}_2^2 \supset C = \{(0,0),(1,1)\}$

minimal distance $d = 2$

The Principle of Error-Correction

To make it easier to understand,

Suppose a sphere with radius e with a member of C in the center, there is no intersection.



When received code enters into one of those spheres, the closest member of C in the center of that sphere is decoded as a sent code.

maximum-likelihood decoding

- Stochastically, e errors can be corrected.
- ★ When d is an even number, $(d-2)/2$ errors can be corrected.

Question $\mathbb{F}_q^n \supset C$

Construct a subset C of which a Hamming distance between 2 arbitrary points in C is most far.

Hamming code, BCH code, RS code, Golay code, ・・・・