Global Focus on Knowledge Lecture Series

Finance and Math

The 10th Lecture

Ito Calculus and Finance

Markoff Process

Discussions on Mathematics in National Newspaper

全國紙上數學談話會

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1077. Markoff過程ヲ定メル微分方程式　伊藤清(1252)　Kiyoshi Ito

Differential Equation That Express Markoff Process

[illegible] [illegible]政賀(1403)

1079. Wimann ノ定理ニツイテ　有馬喜八郎(1406)

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1077. Markoff 過程ヲ定メル微分方程式

伊藤 清（内閣統計局）

ハシガキ

(I) 有限個ノ可能ナ場合 $a_1, a_2, \cdots\cdots a_m$ ヲ有シ、自然数ヲ径数トスル simple markoff process $x_1, x_2, \cdots\cdots$ ニ関シテ、多クノ遷移確率ヲ考ヘルコトガ出来ル。例ヘバ $x_k = a_i$ ナル條件ノ下ニ於ケル $x_l = a_j$ ノ確率、或ハ $x_1 = a_{i_1}, x_2 = a_{i_2}, \cdots\cdots, x_n = a_{i_n}$ ナル條件ノ下ニ於ケル $x_{n+1} = a_{i_{n+1}}$ トナル確率等々。シカシ乍ラソレ等ハ結局 $x_k = a_i$ ノ時ノ $x_{k+1} = a_j$ ナル確率 $p_{ij}^{(k)}$ $(k = 1, 2, \cdots\cdots; i, j = 1, 2, \cdots\cdots m)$ ニ帰着セラレル。コレハ Kolmogoroff ノ本(*1) ニモ書イテアル。以後コレヲ基本的ナ遷移確率ト呼ブ。

更ニ可能ナ場合ガ有限デナクトモ、例ヘバ実数ヲ以テ標識ヅケラレル時ニハ、同ジコトガイヘルノハ云フマデモナイ。

併シナガラ径数ガ自然数デナクテ、実数ノ場合即チ continuous parameter ニ依存スル markoff process ニ於テ、上ノコトハ如何ニナルカトイフコトハソレ程簡単デハナイ。(*2)

更ニ一般ニ可能ナ場合ガ実数ニヨリ標識付ケラレ、且ツ continuous parameter ニ依存スル simple markoff

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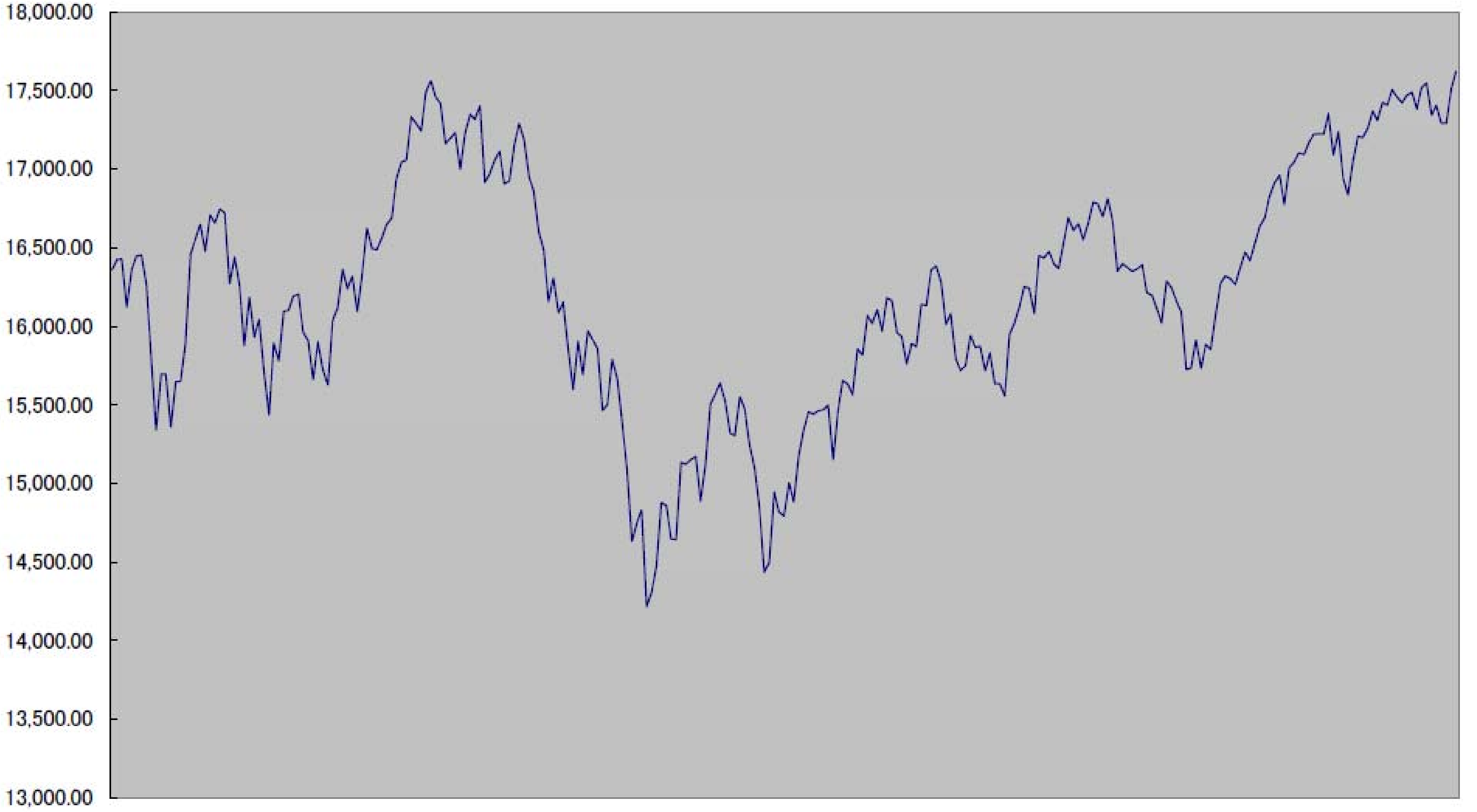
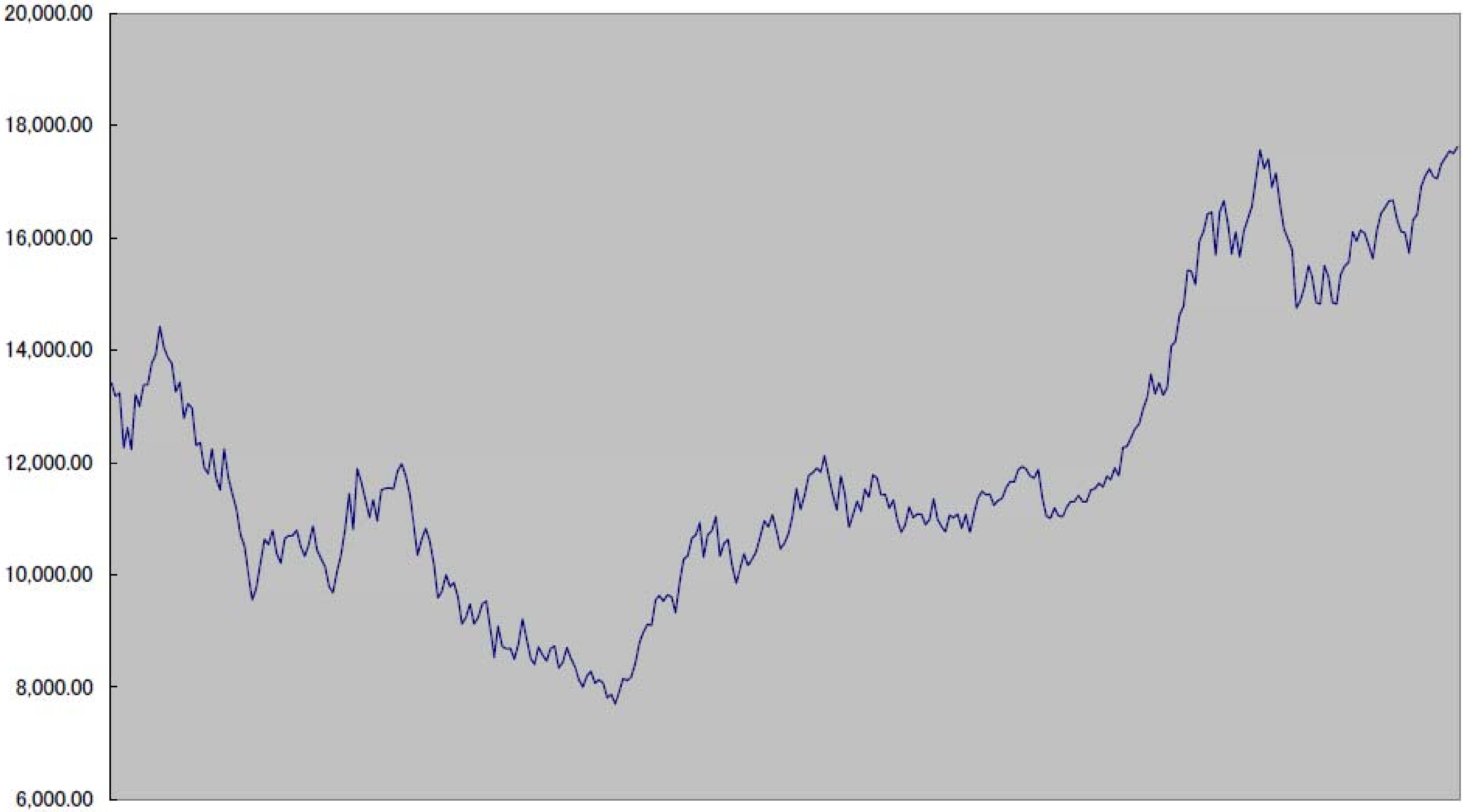
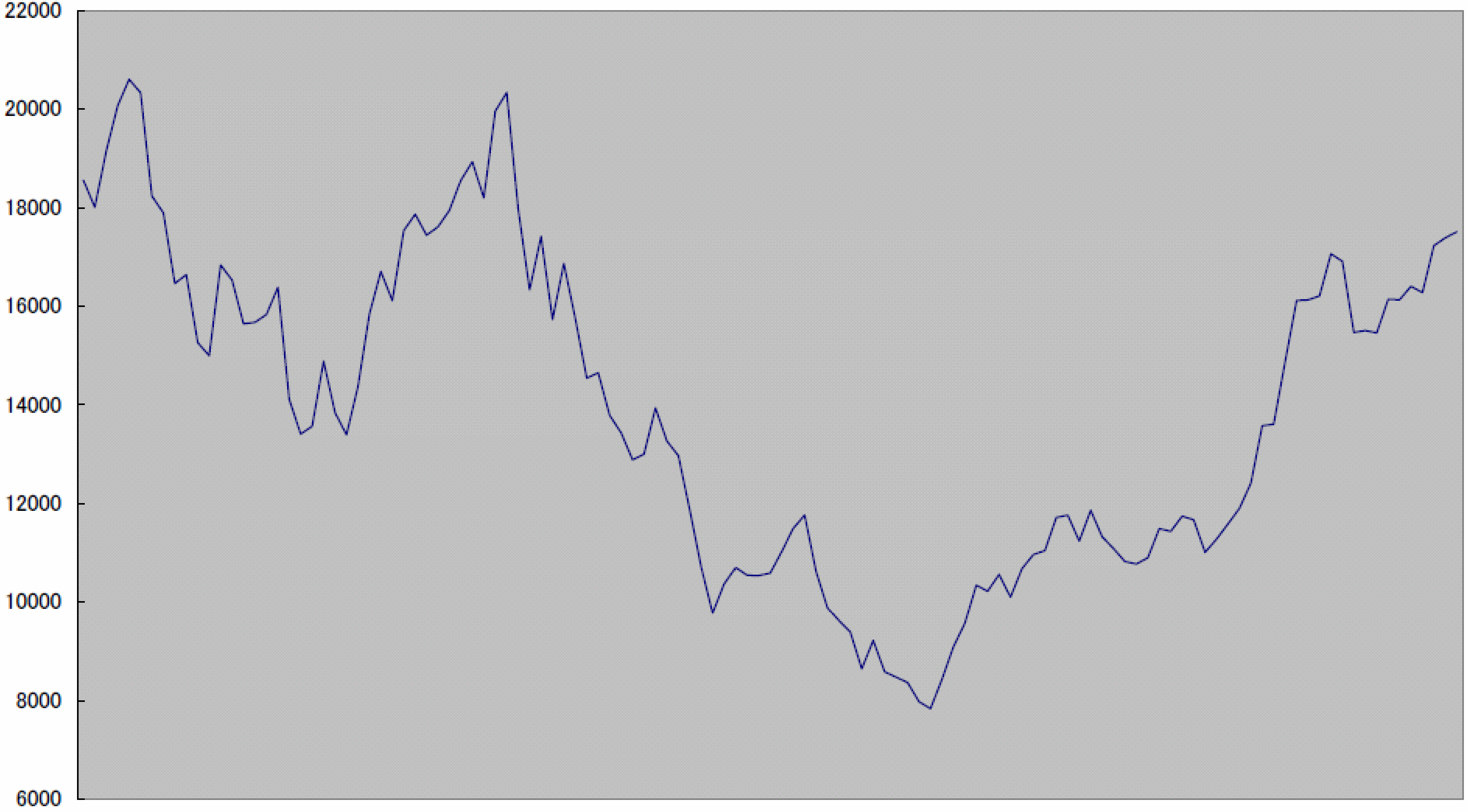
Nikkei Daily Average 2006-2007



Nikkei Weekly Average 2001-2007



Nikkei Monthly Average 1997-2007



Probability Process Model

Dominated by coincidence, and changes every moment

Markoff Process Model

The most important probability process model

Markoff Process

1. time : discrete   space : discrete
2. time : discrete   space : continuous
3. time : continuous   space : discrete
4. time : continuous   space : continuous

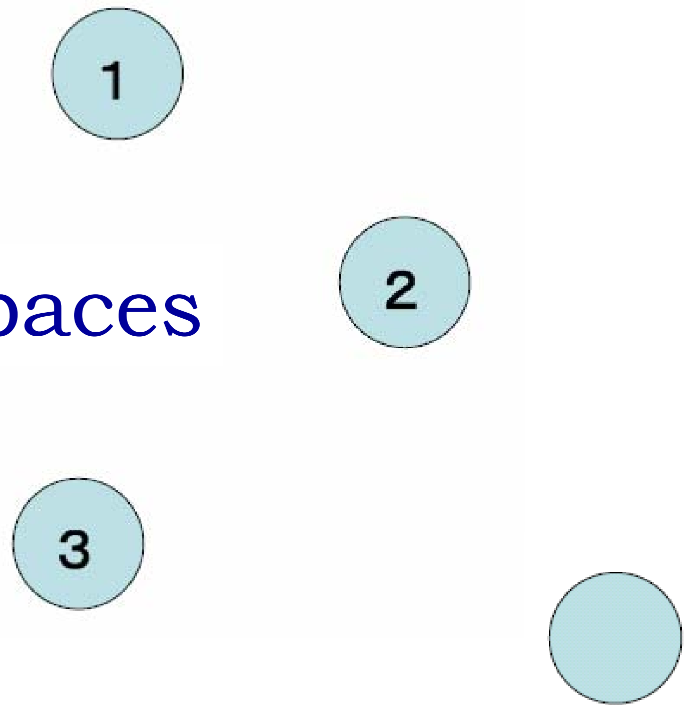
When it moves by jumping

When it moves continuously

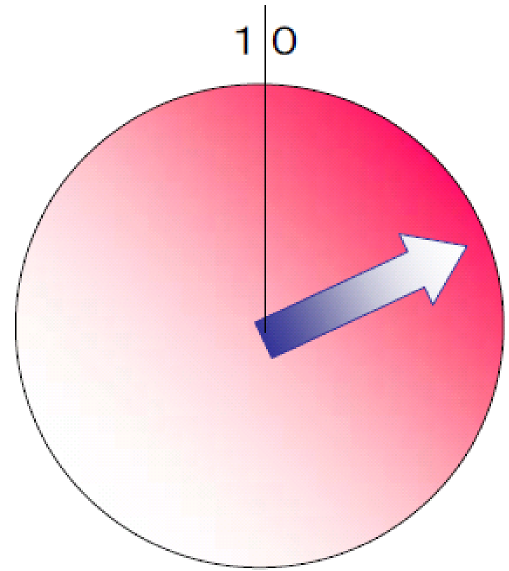
1. time : discrete   space : discrete

solo sugoroku

Solo Sugoroku



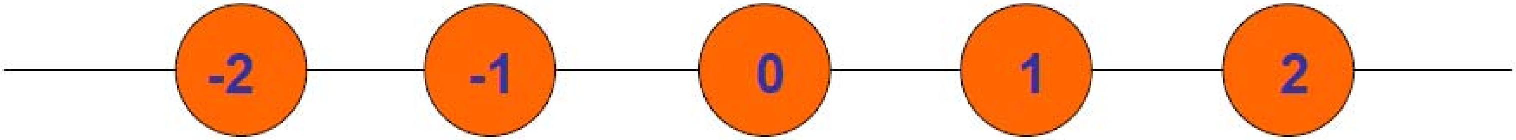
spaces



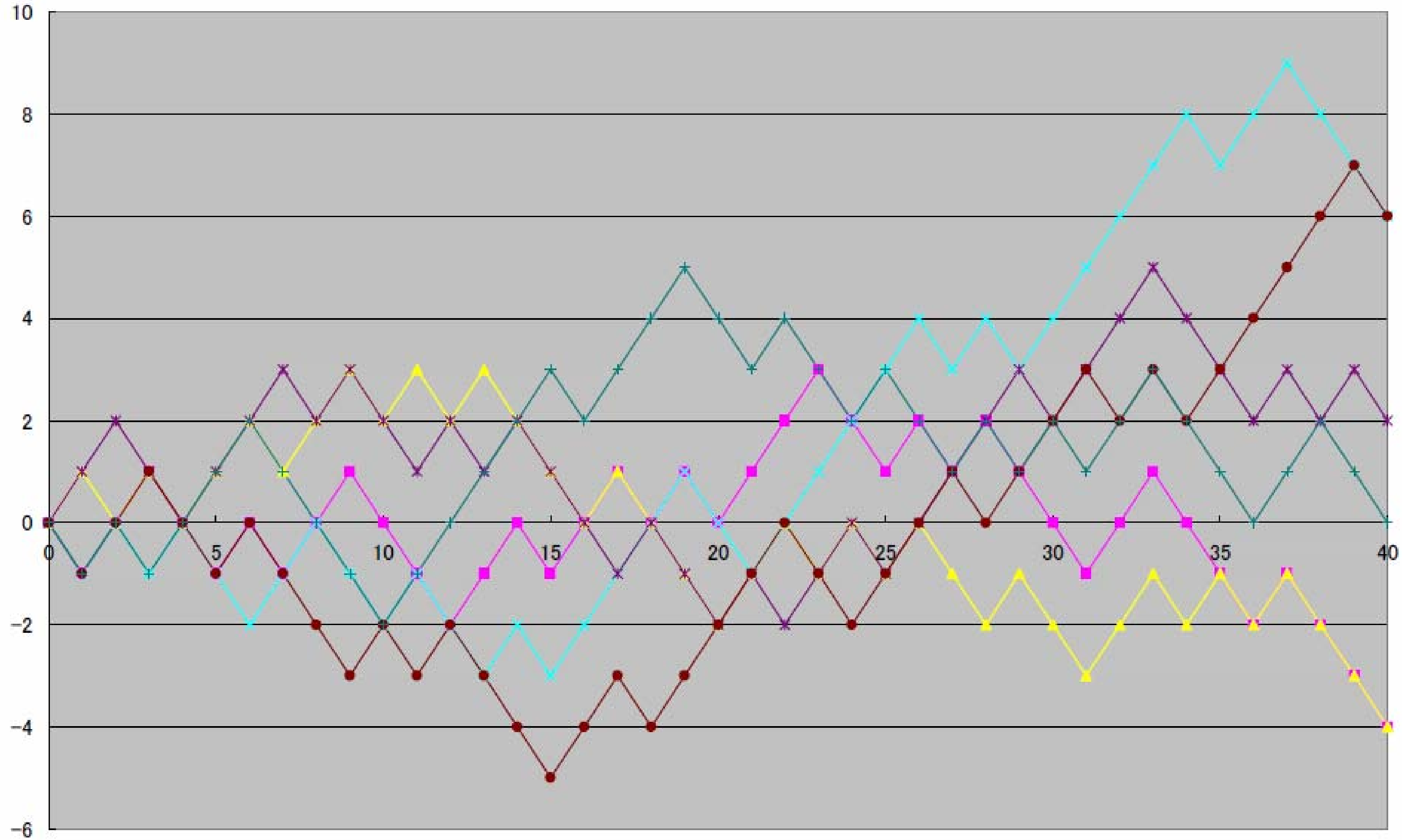
spinning disc

If 0.0-0.38, move to 2.
If 0.38-0.9, move to 5.
If 0.9-1.0, do not move.

Random Walk



If 0.0-0.5, move 1 space to right.
If 0.5-1.0, move 1 space to left.



10
8
6
4
2
0
−2
−4
−6
0
5
10
15
20
25
30
35
40

An example of application : length of queue

"Logic of a Waiting Queue"

How many cash registers are needed in supermarket?

How many cash registers should be running in supermarket at a certain time?

The more cash registers ran, the shorter queue becomes, but employment costs increase.

The less cash registers ran, the longer queue becomes. Customers stop coming, and sales decline.

How many cash registers are most appropriate?

The Simplest Model

A cash register: Serves for one customer in a unit of time

At a certain time-zone, one or two customers come to a cash register at a unit of time.

Probability of one or two customers come

$$p,\ q\ (\ p, q > 0,\quad p + q < 1\ )$$

What is Understood

If 2 cash register run, there is no queue.

When the average number of customers is $p + 2q > 1$,
and if a cash register running is only one, queue gets long.

When $p + 2q < 1$
1 cash register : How long would queue be?

Solo Sugoroku

Spaces: $n = 0, 1, 2, \ldots$

Number of people waiting

space n (n is not 0)

$\Rightarrow$ possibility q $n+1$

possibility p n

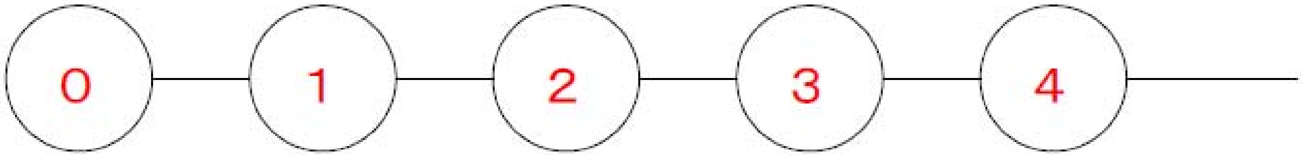
possibility $1-p-q$ $n-1$

space 0

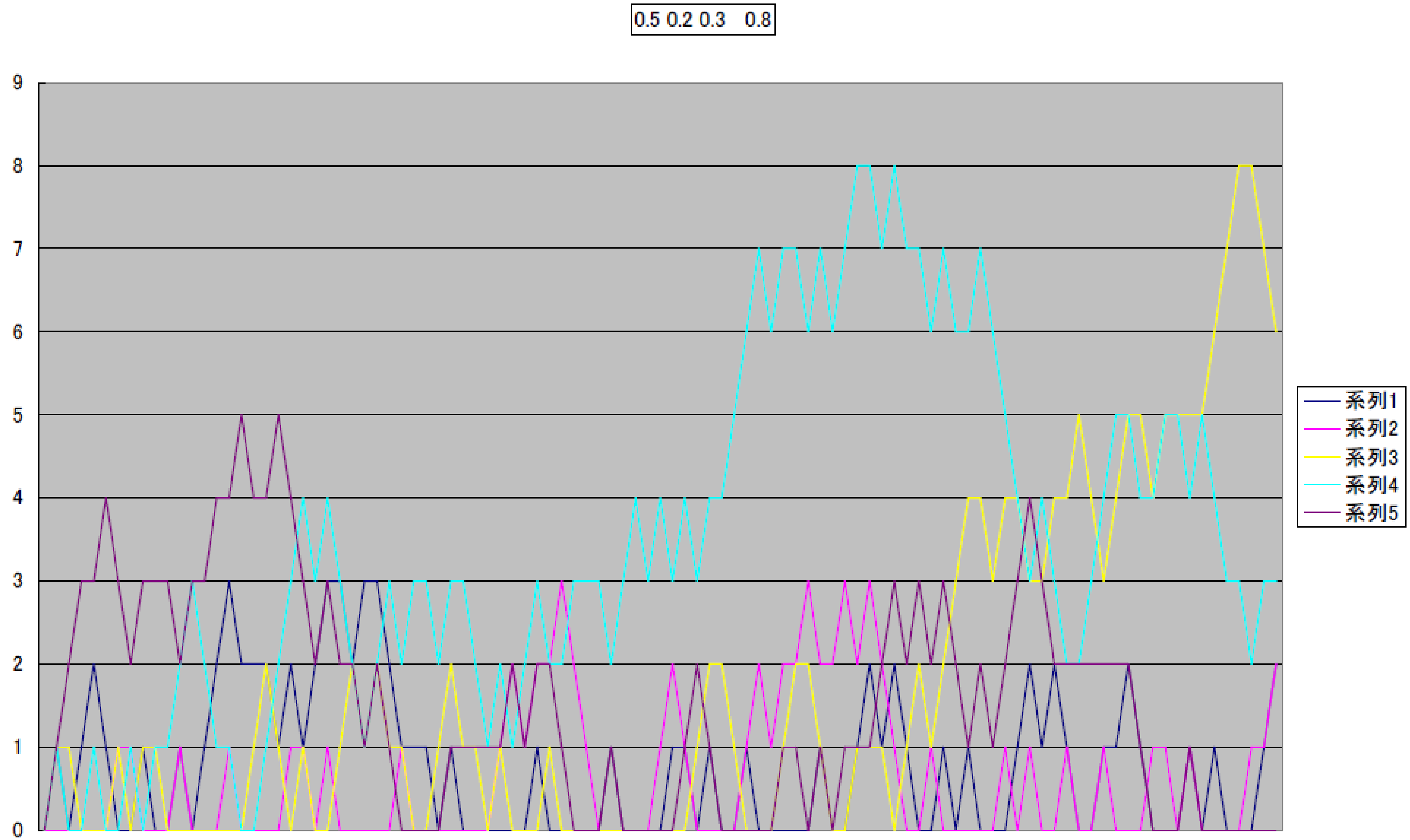
$\Rightarrow$ possibility q 1

possibility $1-q$ 0

Waiting Queue Sugoroku

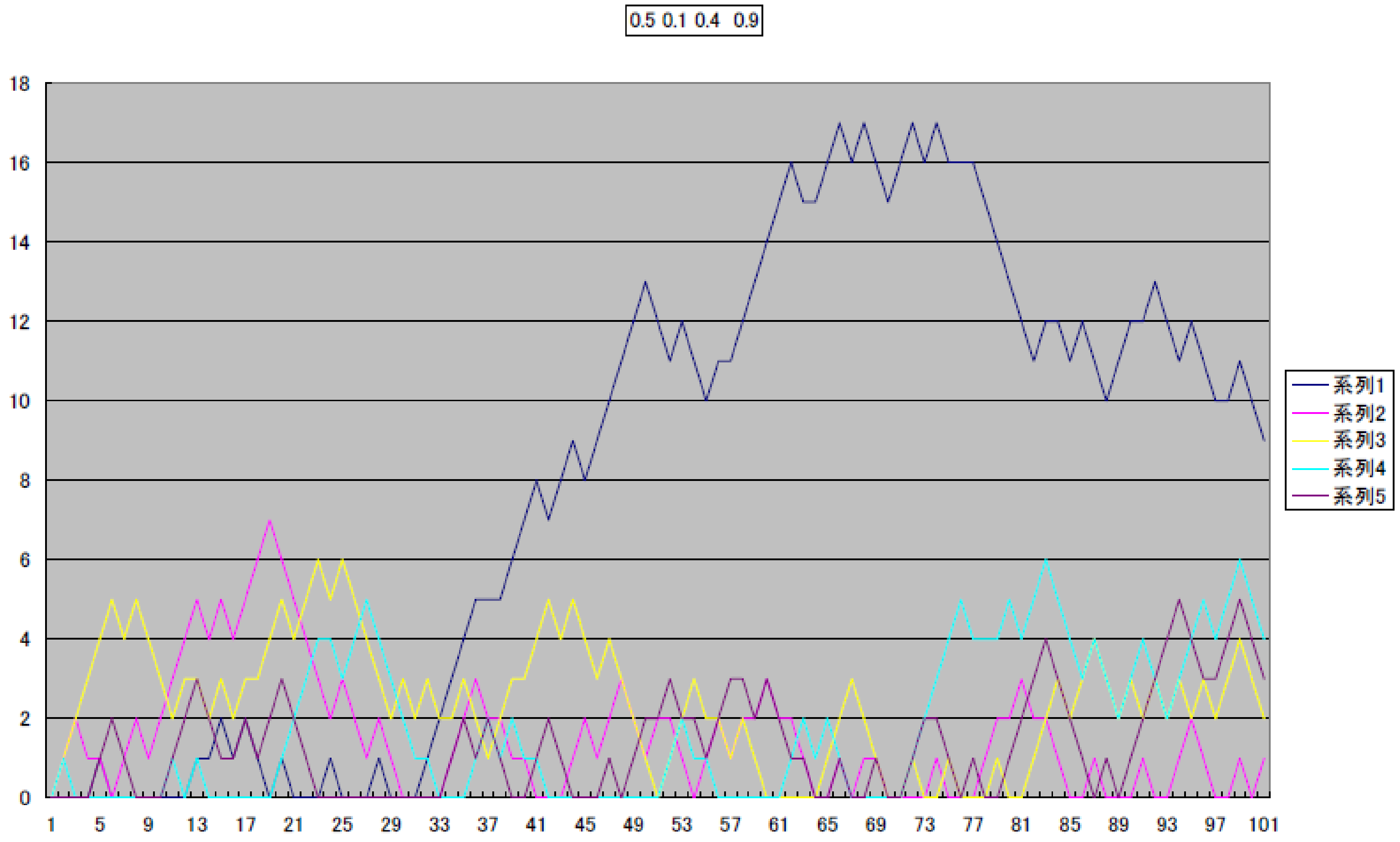


Number of people waiting



0.5 0.2 0.3 0.8
9
8
7
6
5
4
3
2
1
0
系列1
系列2
系列3
系列4
系列5

0.3 0.6 0.1 0.8
3
2
1
0
系列1
系列2
系列3
系列4
系列5



0.5 0.1 0.4 0.9
18
16
14
12
10
8
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21
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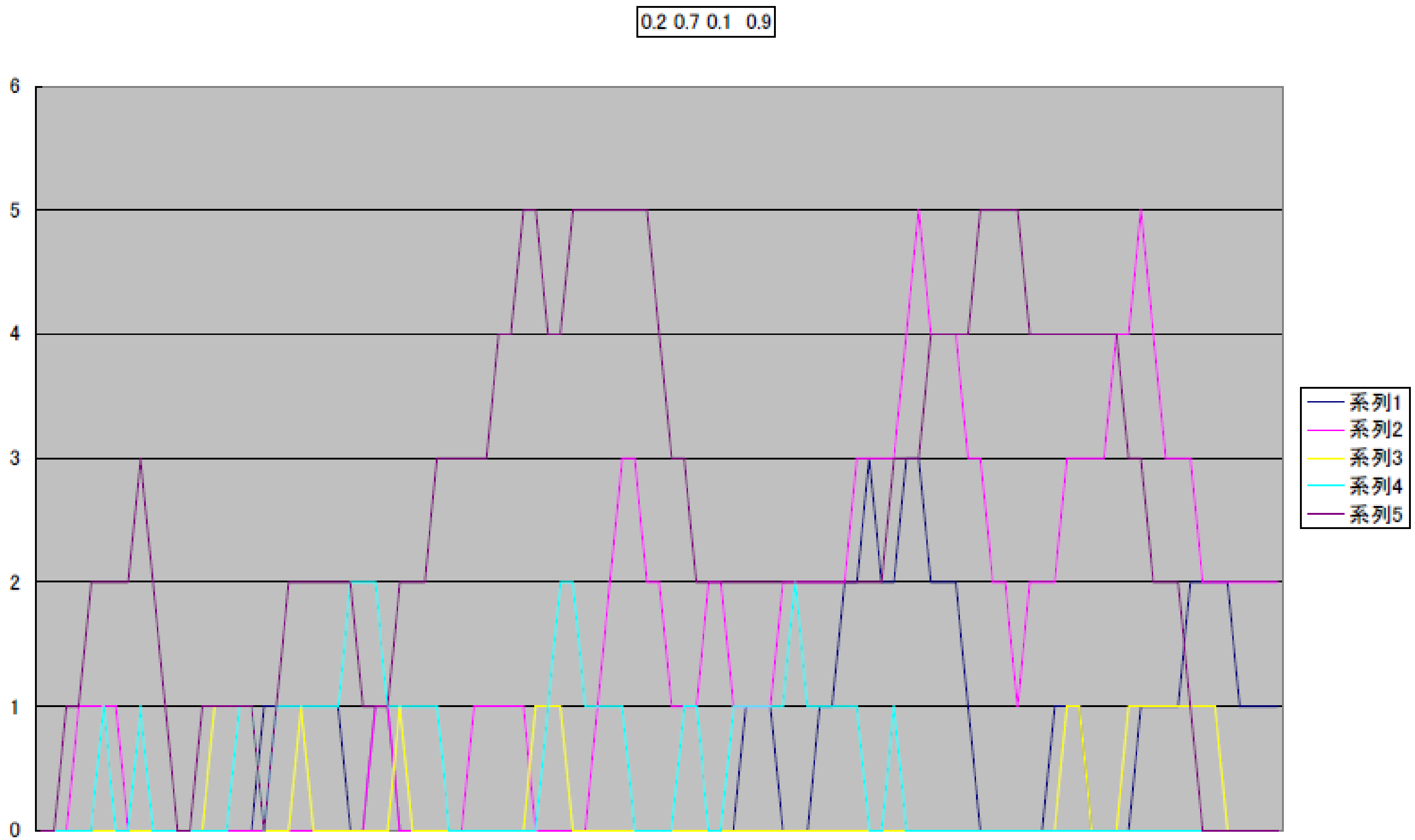
97
101
系列1
系列2
系列3
系列4
系列5



0.2 0.7 0.1 0.9
6
5
4
3
2
1
0
系列1
系列2
系列3
系列4
系列5

Length of a queue after a long time

average length: $M = \frac{q}{1-(p+2q)}$

possibility of more than n people are in a queue: $(\frac{M}{M+1})^n$

$M = 4$ possibility of more than 10 people are in a queue $\geqq 0.1$

$M = 1.5$ possibility of more than 4 people are in a queue $\geqq 0.12$

If there is only one counter, answer can be obtained by a simple calculation.

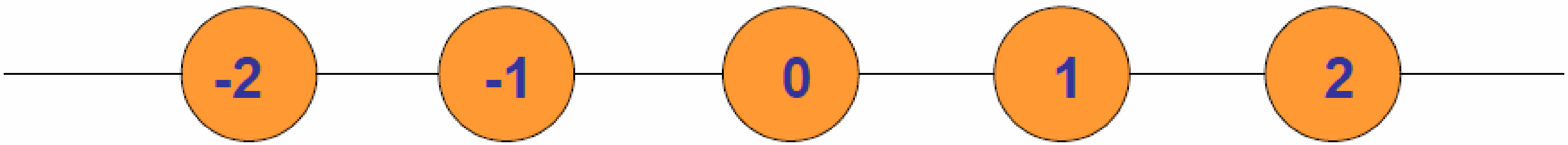
If there are more than 2 counters, generally, it cannot be solved.

Model of diverse waiting lines

It is now used in huge systems

as a basic model for system planning.

Random Walk



$x \to x+1$ possibility of 1/2

$x \to x-1$ possibility of 1/2

Random Walk

recurring formula corresponding to the rule

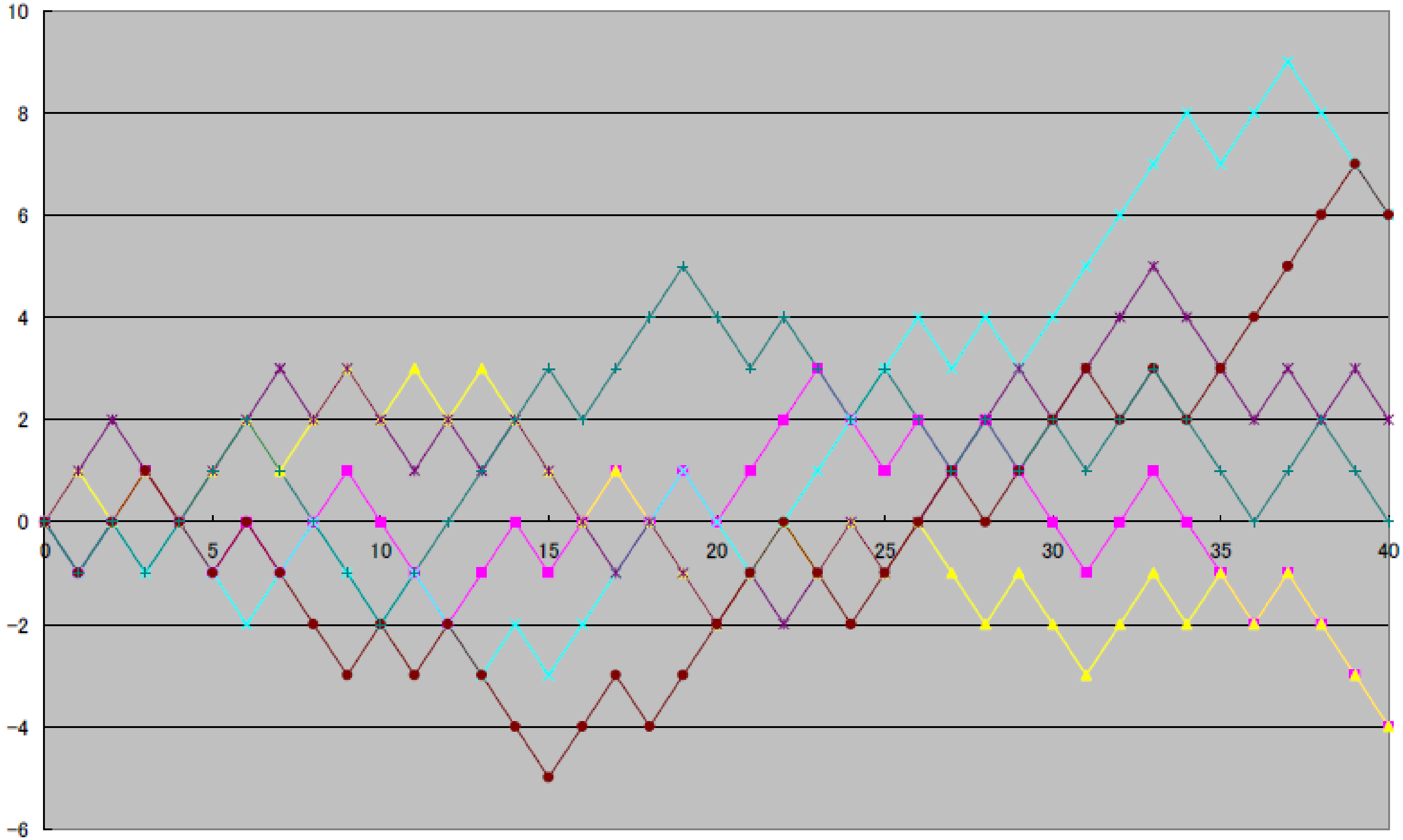
$$u(t+1,x) = \frac{1}{2}(u(t,x+1) + u(t,x-1))$$

difference equation rewritten

$$u(t+1,x) - u(t,x) = \frac{1}{2}(u(t,x+1) + u(t,x-1) - 2u(t,x))$$

equation that average value (statistic value) follows

This equation decides the rule of sugoroku.



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-4
-6
0
5
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15
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25
30
35
40

Generalization

2. time : discrete space : continuous
3. time : continuous space : discrete
4. time : continuous space : continuous

when you move by jumping

easy

when you move continuously

diffusion process

Kolmogorov 1931

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Kolmogorov's idea (1931)

space : line (can be applied to multi-dimension easily)
space : all real numbers

When one starts from x and gets to the space y after dt hours distribution of y is $x + b(x)dt$ in average, variance is $\sigma(x)^2 dt$

dependent on locations!

Diffusion equation that average value follows:

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sigma(x)^2\frac{\partial^2}{\partial x^2}u(t,x) + b(x)\frac{\partial}{\partial x}u(t,x)$$

Kolmogorov (1931)'s Diffusion Equation

rule of sugoroku, description of average value
How pieces actually move is wanted to be described.

stochastic differential equation

Quite a few people must have thought of …

Bachelier, Levy …

The idea that a dice is rolled continuously
is a mathematical paradox. (Doob)

Let us first define "probability integral"
thinking Brownian motion as a basis.

Brownian motion: the extreme of random walk

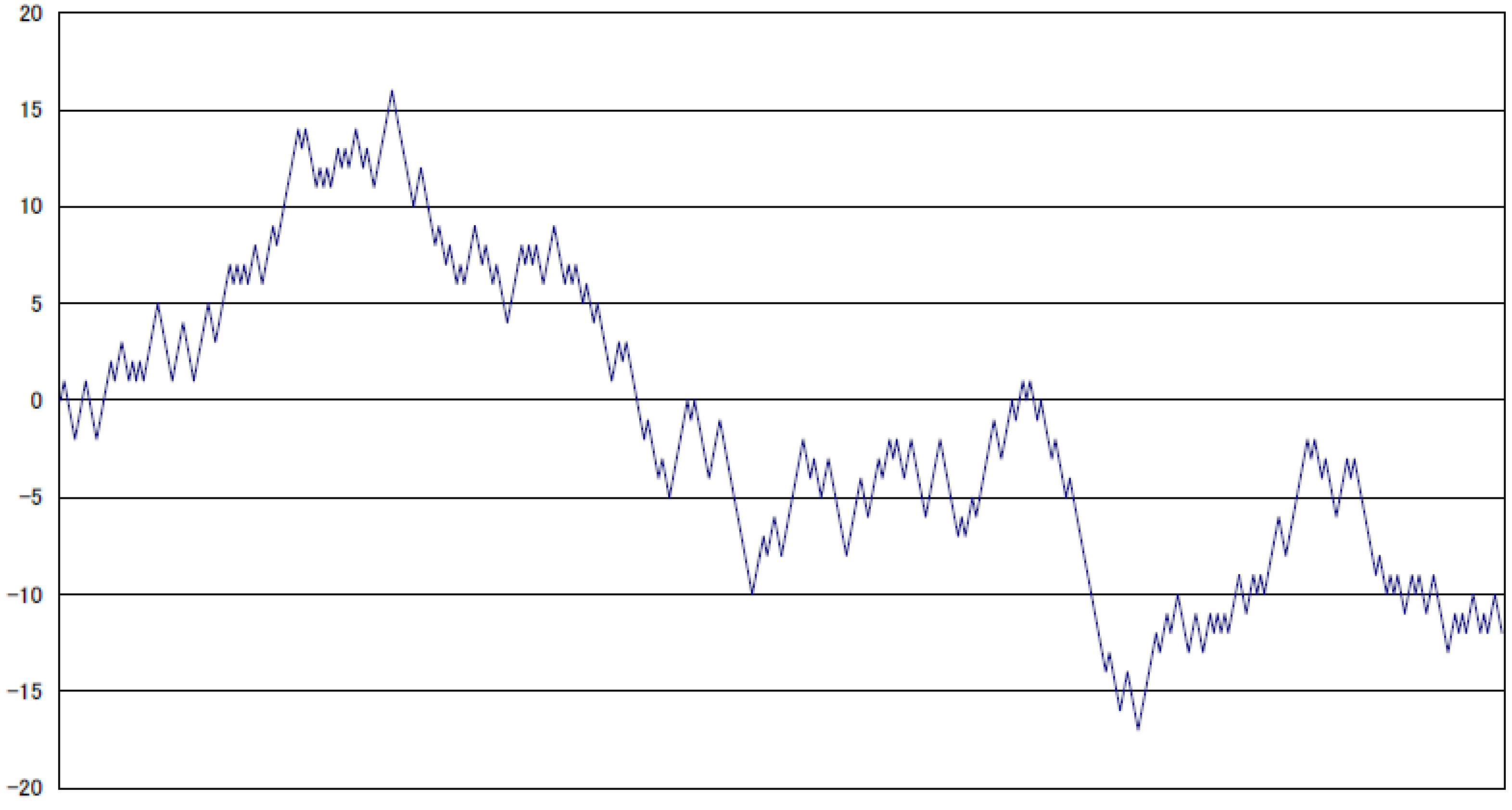
Bachelier 1900 the model of stock market
Einstein 1905 Brownian motion, atomistics

They induced partial differential equation.

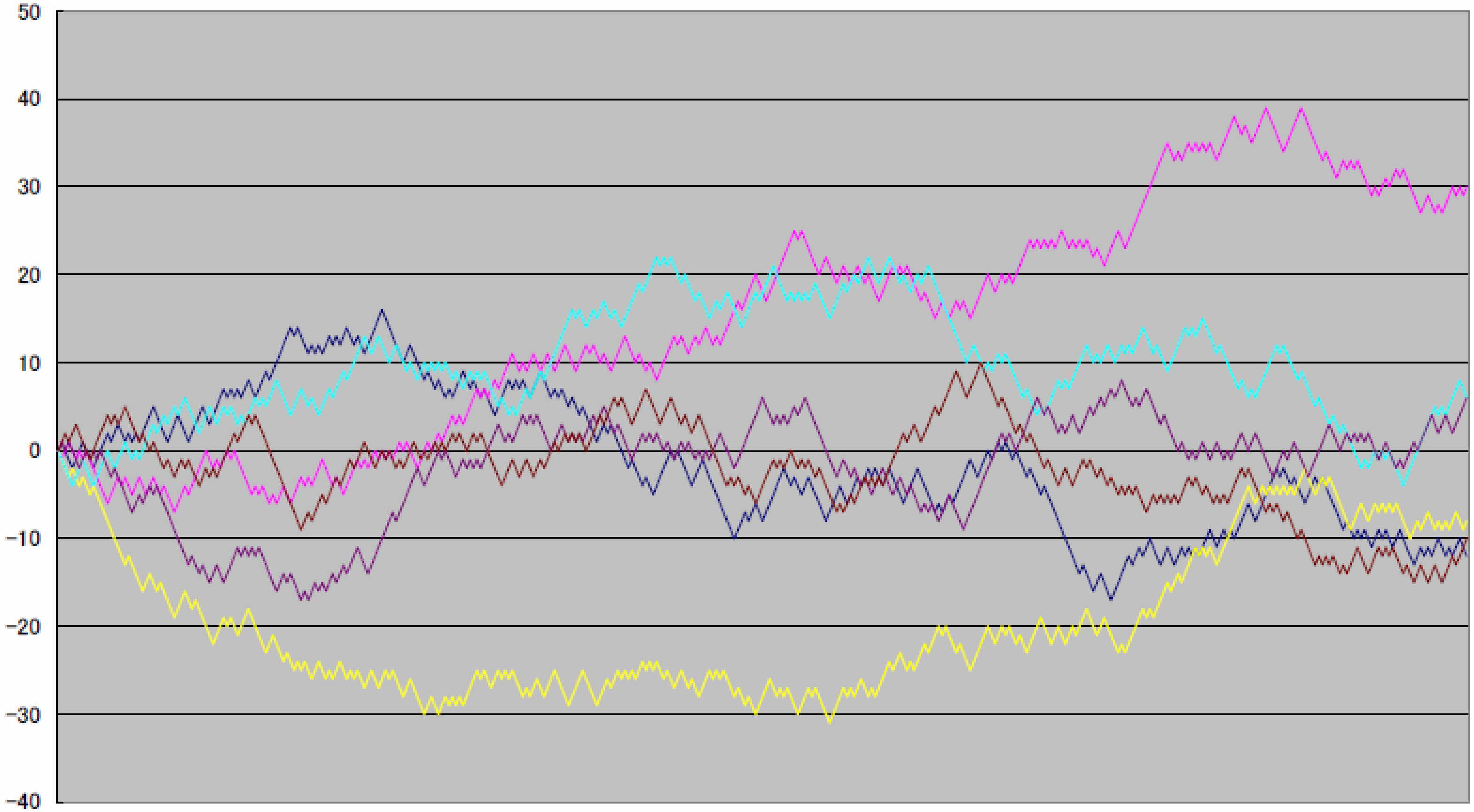
$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x)$$

Wiener 1923: constructed **measure** in functional space
Each orbit of Brownian motion is described.
This evolved from establishment of the measure theory.

Brownian motion



Brownian motion



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Ito 1942

Suppose increase of Brownian motion dB_t as a basis

$$x_t \longrightarrow x_t + \sigma(x_t)dB_t + b(x_t)dt$$

Brownian motion: role of a dice in sugoroku

stochastic differential equation

$$dx_t = \sigma(x_t)dB_t + b(x_t)dt$$

“Differential” is just a formal word here.

Consider it as stochastic integral equation.

$$x_t = x_0 + \int_0^t \sigma(x_s)dB_s + \int_0^t b(x_s)ds$$

What Ito (1942) Has Proved

Definition of probability integration

Existence and uniqueness of answer to stochastic differential equation

Induction of Kolmogorov's diffusion equation

Complex discussion is needed.

1951 Discovery of **Ito's rule**

Kunida-Watanabe (1967) generalization of probability integration

$\Downarrow$

Meyer, Strasbourg School

Finance

Black-Scholes (1973)

The price of European call option

Model

Bond price : constant interest

Stock price: geometrical Brownian motion

dynamic hedging arbitrary

Change portfolio of stock and bond

time after time, and duplicate the option.

Price of Option

Europe call option

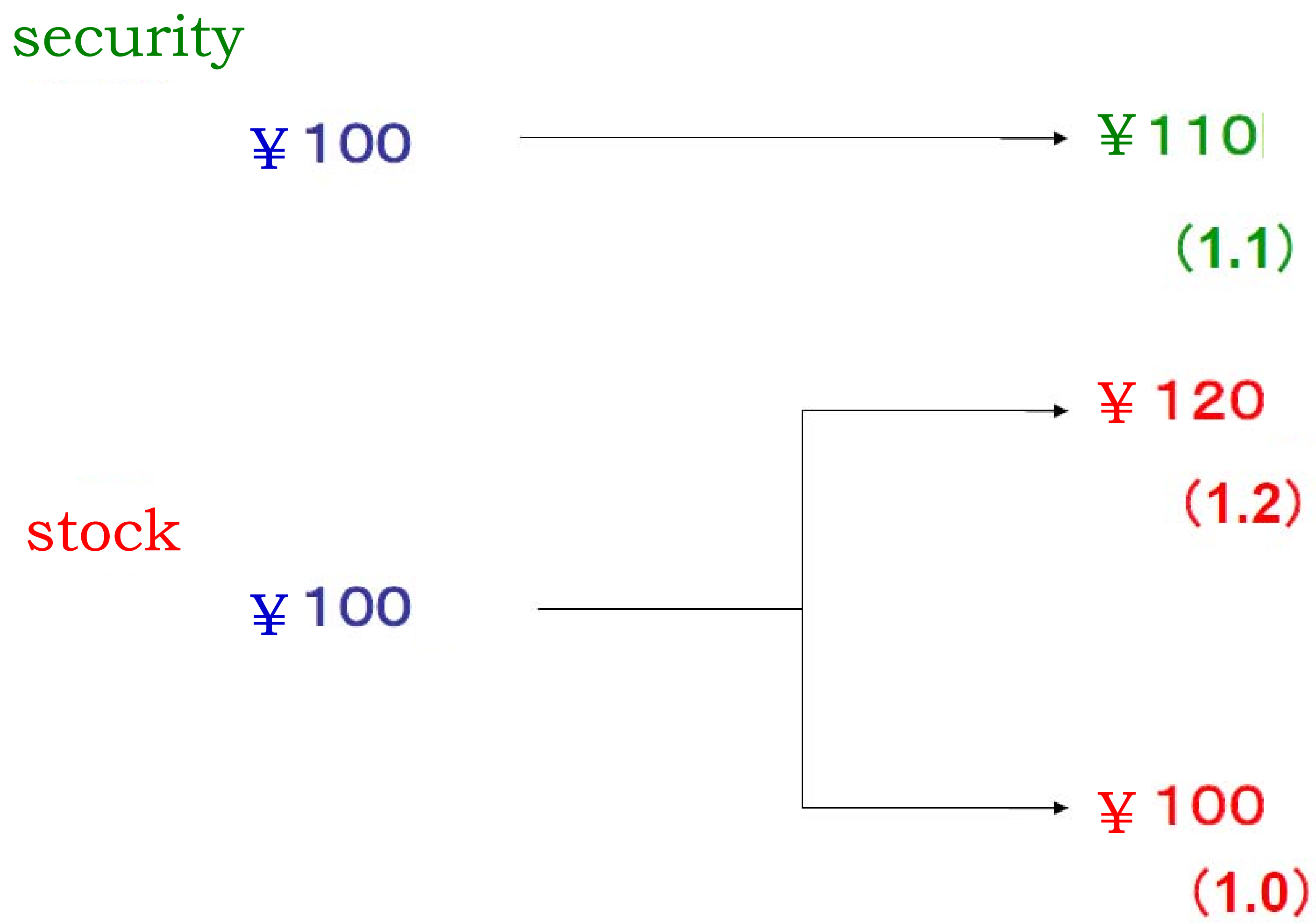
A right to buy a certain amount of

a certain security

on a certain date (date of expiration)

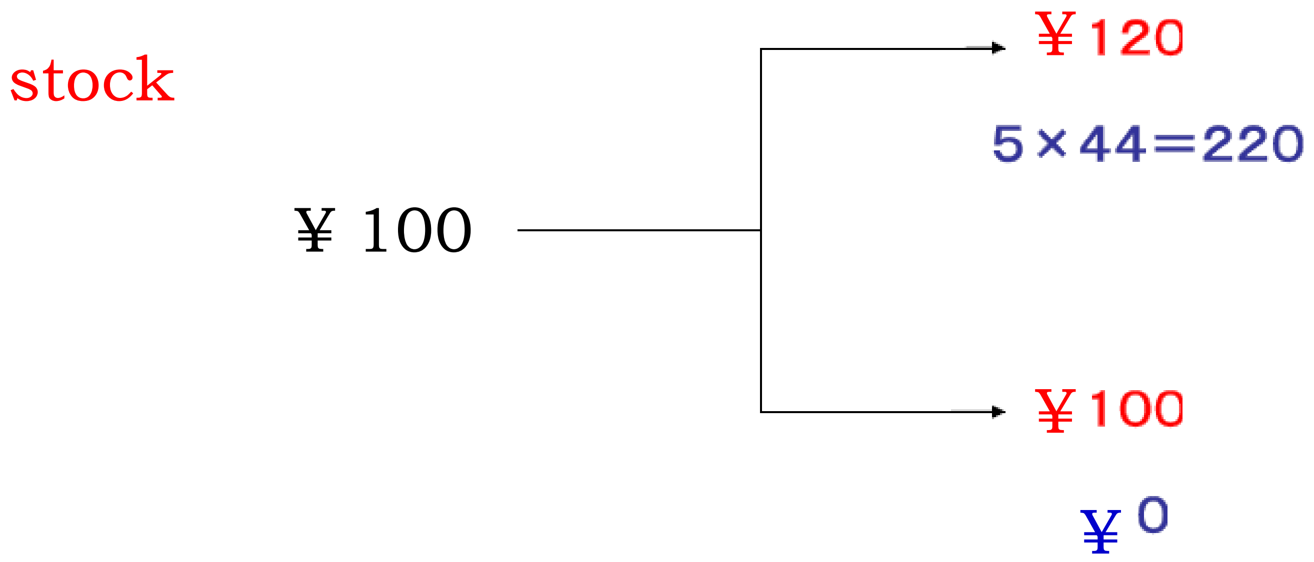
at a certain price (practice price)

This is a right, so there is no necessity to use.


security
¥100
¥110
(1.1)
stock
¥100
¥120
(1.2)
¥100
(1.0)

Stock Call Option

Buy 44 stocks of 1 description at ¥115



Duplication of Option

stock x

security y

(x+y=100)

X=1100, y=-1000

¥ 1.2x + 1.1y

¥ 440

1.2x+1.1y=440

¥ 1.0x + 1.1y

¥ 0

1.0x+1.1y= 0

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Merton（1973）

Justified Black-Scholes' discussion

Rewrote the model in Ito analysis

In BS paper, probability process was assumed in the back.
Only differential equation is expressed in the front.

Portfolio change time after time

⇒ probability integration

Censoring at secondary differentiation

⇒ Ito's rule

Later developments

Harrison-Kreps（1979）

Generalization of a model

Successes after Kunida & Watanabe, Meyer is needed.

Ito’s Representation Theory

Later, more complex models appear in practical business.

(To explain practical phenomena better)