Nonlinear Finite Element Method

29/11/2004

Nonlinear
Finite Element Method

- Lectures include discussion of the nonlinear finite element method.
- It is preferable to have completed “Introduction to Nonlinear Finite Element Analysis” available in summer session.
- If not, students are required to study on their own before participating this course. Reference:Toshiaki.,Kubo. “Introduction: Tensor Analysis For Nonlinear Finite Element Method” (Hisennkei Yugen Yoso no tameno Tensor Kaiseki no Kiso),Maruzen.
- Lecture references are available and downloadable at http://www.sml.k.u-tokyo.ac.jp/members/nabe/lecture2004 They should be posted on the website by the day before scheduled meeting, and each students are expected to come in with a copy of the reference.
- Lecture notes from previous year are available and downloadable, also at http://www.sml.k.u.tokyo.ac.jp/members/nabe/lecture2003 You may find the course title, ”Advanced Finite Element Method” but the contents covered are the same I will cover this year.
- I will assign the exercises from this year, and expect the students to hand them in during the following lecture. They are not the requirements and they will not be graded, however it is important to actually practice calculate in deeper understanding the finite element method.
- For any questions, contact me at nabe@sml.k.u-tokyo.ac.jp

Nonlinear Finite Element Method Lecture Schedule

1. 10/ 4 Finite element analysis in boundary value problems and the differential equations
2. 10/18 Finite element analysis in linear elastic body
3. 10/25 Isoparametric solid element (program)
4. 11/ 1 Numerical solution and boundary condition processing for system of linear equations（with exercises）
5. 11/ 8 Basic program structure of the linear finite element method(program)
6. 11/15 Finite element formulation in geometric nonlinear problems(program)
7. 11/22 Static analysis technique、hyperelastic body and elastic-plastic material for nonlinear equations (program)
8. 11/29 Exercises on Lecture7
9. 12/ 6 Dynamic analysis technique and eigenvalue analysis in the nonlinear equations
10. 12/13 Structural element
11. 12/20 Numerical solution— skyline method、iterative method for the system of linear equations
12. 1/17 ALE finite element fluid analysis
13. 1/24 ALE finite element fluid analysis

Hyperelastic Body

First, let us clarify the symbols that are used in this material to define certain quantities.

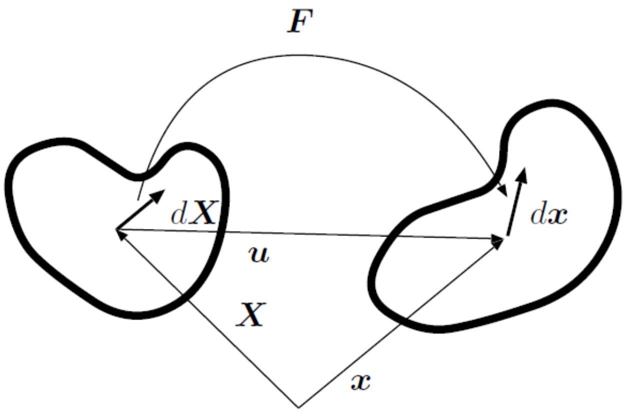

図 1: 変形勾配の概念図

$$
\left\{
\begin{array}{lll}
\boldsymbol{X}, \boldsymbol{x} & : & \text{変形前後の物質点位置ベクトル} \\
\boldsymbol{u} & : & \text{変位 } (= \boldsymbol{x} - \boldsymbol{X}) \\
\boldsymbol{F} & : & \text{変形勾配テンソル} \\
\boldsymbol{C} & : & \text{右 Cauchy–Green 変形テンソル} \\
\boldsymbol{B} & : & \text{左 Cauchy–Green 変形テンソル} \\
\boldsymbol{E} & : & \text{Green-Lagrange 歪テンソル} \\
\boldsymbol{T} & : & \text{Cauchy 応力テンソル} \\
\boldsymbol{\Pi} & : & \text{第 1 Piola–Kirchhoff 応力テンソル} \\
\boldsymbol{S} & : & \text{第 2 Piola–Kirchhoff 応力テンソル}
\end{array}
\right. \tag{1}
$$

$$\boldsymbol{F} \equiv \frac{\partial x_i}{\partial X_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2}$$

$$\boldsymbol{C} \equiv \boldsymbol{F}^T \cdot \boldsymbol{F} \tag{3}$$

$$\boldsymbol{B} \equiv \boldsymbol{F} \cdot \boldsymbol{F}^T \tag{4}$$

$$\boldsymbol{E} \equiv \frac{1}{2} (\boldsymbol{C} - \boldsymbol{I}) \tag{5}$$

$$\boldsymbol{\Pi} \equiv J \boldsymbol{F}^{-1} \cdot \boldsymbol{T} \tag{6}$$

$$\boldsymbol{S} \equiv J \boldsymbol{F}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{F}^{-T} \tag{7}$$

Provided that $\boldsymbol{e}_i$ represents a base vector, $\otimes$ represents the tensor product, and $J = \det \boldsymbol{F}$

Incompressible Hyperelastic Body 1

- Hyperelastic body is defined as a substance,in which the elastic potential function W exists. The elastic potential function gives a conjugate stress component by taking its differentials of the components deformation or a strain.

$$S_{ij} = \frac{\partial W}{\partial E_{ij}} \tag{8}$$

- $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$ hence,

$$S_{ij} = 2\frac{\partial W}{\partial C_{ij}} \tag{9}$$

- Since W requires the objectivity in the scalar, which is expressed as a function of principal value in $\boldsymbol{C}$. In addition, the principle value makes a function of principle invariants defined as following so, W and $\boldsymbol{C}$ can be expressed in the function of principle invariants as well.

$$I_C \equiv \mathrm{tr}\boldsymbol{C} \tag{10}$$

$$II_C \equiv \frac{1}{2}\left\{(\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2)\right\} \tag{11}$$

$$III_C \equiv \det \boldsymbol{C} \tag{12}$$

- Thus,

$$S_{ij} = 2\left(\frac{\partial W}{\partial I_C}\frac{\partial I_C}{\partial C_{ij}} + \frac{\partial W}{\partial II_C}\frac{\partial II_C}{\partial C_{ij}} + \frac{\partial W}{\partial III_C}\frac{\partial III_C}{\partial C_{ij}}\right) \tag{13}$$

Incompressibility Hyperelastic Body 2

- Furthermore,

$$\frac{\partial I_C}{\partial C_{ij}} = \delta_{ij} \tag{14}$$

$$\frac{\partial II_C}{\partial C_{ij}} = I_C \delta_{ij} - C_{ij} \tag{15}$$

$$\frac{\partial III_C}{\partial C_{ij}} = III_C (\boldsymbol{C}^{-1})_{ij} \tag{16}$$

Based on above,

$$S_{ij} = 2\left\{ \left(\frac{\partial W}{\partial I_C} + \frac{\partial W}{\partial II_C} I_C \right) \delta_{ij} - \frac{\partial W}{\partial II_C} C_{ij} + \frac{\partial W}{\partial III_C} III_C (\boldsymbol{C}^{-1})_{ij} \right\} \tag{17}$$

- Apparently, the direction of principle axis in $\boldsymbol{S}$ and $\boldsymbol{C}$ coincide,
- Rewrite the above into Cauchy stress.

$$T_{kl} = \frac{2}{J}\left\{ \left(\frac{\partial W}{\partial II_B} II_B + \frac{\partial W}{\partial III_B} III_B \right) \delta_{kl} + \frac{\partial W}{\partial I_B} B_{kl} - \frac{\partial W}{\partial II_B} III_B (\boldsymbol{B}^{-1})_{kl} \right\} \tag{18}$$

- Still, the direction of principle axis $\boldsymbol{T}$ and $\boldsymbol{B}$ coincide.

Incompressible Hyperelastic Body 3

- The high polymer materials possess characteristics to stay constant even under the extensive deformation, and generally, such substance are modeled in supposition of incompressibility.
- By adding equal force to the incompressible materials, the inner stress occurs without any change in volume being observed.
- In a case where there is only the hydrostatic pressure acting on the incompressible substance, the inner stress matches with the hydrostatic pressure while there is no deformation being observed.
- This inner stress is called a nondeterministic stress for which cannot be determined by the history of movement the substance points follow.
- In analyzing an incompressible substance by modeling the hyperelastic body, we need to take an nondeterministic stress (indefinite hydrostatic pressure)as independent variables.
- Here, consider $III_C = III_B = 1$ and $J = 1$,

$$T_{kl} = -p\delta_{ij} + 2\left\{ \left(\frac{\partial W}{\partial II_B} II_B + \frac{\partial W}{\partial III_B} \right) \delta_{kl} + \frac{\partial W}{\partial I_B} B_{kl} - \frac{\partial W}{\partial II_B} (\boldsymbol{B}^{-1})_{kl} \right\} \tag{19}$$

- Where p represent the indefinite hydrostatic pressure determined by the boundary condition.
- Again, rewrite the equation in the second Piola-Kirchhoff stress, then given by,

$$S_{ij} = -p(\boldsymbol{C}^{-1})_{ij} + 2\left\{ \left(\frac{\partial W}{\partial I_C} + \frac{\partial W}{\partial II_C} I_C \right) \delta_{ij} - \frac{\partial W}{\partial II_C} C_{ij} + \frac{\partial W}{\partial III_C} (\boldsymbol{C}^{-1})_{ij} \right\} \tag{20}$$

Mooney-Rivlin Material 1

- Often, Mooney-Rivlin material is applied for the elasticity potential function *W* of incompressible hyperelastic body. $$W^M \equiv c_1(I_C - 3) + c_2(II_C - 3) \tag{21}$$

where *c*1 and *c*2 are invariables defined by the substance being used.

The second Piola-Kirchhoff stress can be gained by adopting Mooney-Rivlin material.

$$S_{ij} = -p(\boldsymbol{C}^{-1})_{ij} + 2\Big\{(c_1 + c_2 I_C)\delta_{ij} - c_2 C_{ij}\Big\} \tag{22}$$

- Therefore, when there is no external force present and no deformation is made, represented as *Cij* = *δij*,

$$T_{ij} = S_{ij} = 0 \tag{23}$$

$$S_{ij} = -p\delta_{ij} + (2c_1 + 4c_2)\delta_{ij} \tag{24}$$

p is allowed to have the initial value 2*c*1 + 4*c*2.

- In order to provide a resolution to this inconvenience, a modified model of *WM* is often considered.

$$W_R^M \equiv c_1(\widetilde{I}_C - 3) + c_2(\widetilde{II}_C - 3) \tag{25}$$

Where,

$$\widetilde{I}_C \equiv \frac{I_C}{III_C{}^{\frac{1}{3}}} \tag{26}$$

$$\widetilde{II}_C \equiv \frac{II_C}{III_C{}^{\frac{2}{3}}} \tag{27}$$

Mooney-Rivlin Model 2

- $\widetilde{I}_C$, $\widetilde{II}_C$ are called reduced invariants.
- The second Piola-Kircchoff stress can be obtained based on W_R^M ,

$$\frac{\partial W_R^M}{\partial I_C} = \frac{\partial W_R^M}{\partial \widetilde{I}_C}\frac{\partial \widetilde{I}_C}{\partial I_C} = c_1 III_C^{-\frac{1}{3}} \tag{28}$$

$$\frac{\partial W_R^M}{\partial II_C} = \frac{\partial W_R^M}{\partial \widetilde{II}_C}\frac{\partial \widetilde{II}_C}{\partial II_C} = c_2 III_C^{-\frac{2}{3}} \tag{29}$$

$$\frac{\partial W_R^M}{\partial III_C} = \frac{\partial W_R^M}{\partial \widetilde{I}_C}\frac{\partial \widetilde{I}_C}{\partial III_C} + \frac{\partial W_R^M}{\partial \widetilde{II}_C}\frac{\partial \widetilde{II}_C}{\partial III_C} = -\frac{1}{3}c_1 I_C III_C^{-\frac{4}{3}} - \frac{2}{3}c_2 II_C III_C^{-\frac{5}{3}} \tag{30}$$

Therefore,

$$S_{ij} = -p(C^{-1})_{ij} + 2\left\{(c_1 + c_2 I_C)\delta_{ij} - c_2 C_{ij} + \left(-\frac{1}{3}c_1 I_C - \frac{2}{3}c_2 II_C\right)(\boldsymbol{C}^{-1})_{ij}\right\} \tag{31}$$

- Under the absence of deformation.

$$T_{ij} = S_{ij} = 0 \tag{32}$$

$$S_{ij} = -p\delta_{ij} \tag{33}$$

Hence, there should be no such inconvenience as p to possess an initial value.

Mooney-Rivlin Model 3

- Reduced invariants contain a physical signification explained in the following.
- In considering incompressible materials, it is appropriate to handle the elastic deformation in isochore deformation and enhanced deformation in separately. To do so, we define the part of isochore deformation in tensor *F* by deformation gradient.

$$\widetilde{\boldsymbol{F}} = J^{-\frac{1}{3}}\boldsymbol{F} \qquad (34)$$

- Here $\boldsymbol{F}$ is called Flory's deformation gradient tensor, having det $\boldsymbol{F} = 1$ for arbitrary deformation.
- Modified right Cauchy-Green deformation tensor $\boldsymbol{C}$ can be defined by,

$$\widetilde{\boldsymbol{C}} = \widetilde{\boldsymbol{F}}^T \cdot \widetilde{\boldsymbol{F}} \qquad (35)$$

- Since reduced invariants takes the first invariant and the second invariant of C, we obtain $\widetilde{I}_C = 3, \widetilde{II}_C = 3$ simple pulling with finite deformation.

Young's Modulus and Shearing Modulus in Infinitesimal Deformation 1

- To verify that hyperelastic body concord with linear elastic body by taking arbitrary c_1 and c_2 under infinitesimal displacement, consider a simple pulling deformation in the figure below.

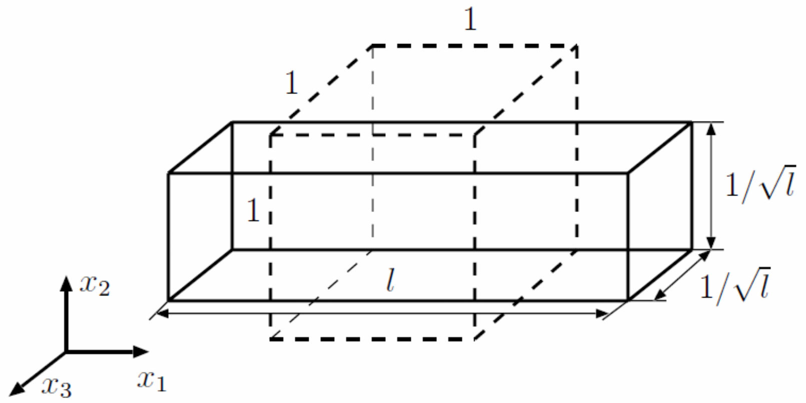

図 3: 単純引張変形

Now, $\boldsymbol{F}$, $\boldsymbol{B}$ and II_B are respectively,

$$\boldsymbol{F} = \begin{bmatrix} l & 0 & 0 \\ 0 & 1/\sqrt{l} & 0 \\ 0 & 0 & 1/\sqrt{l} \end{bmatrix} \tag{38}$$

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = \begin{bmatrix} l^2 & 0 & 0 \\ 0 & 1/l & 0 \\ 0 & 0 & 1/l \end{bmatrix} \tag{39}$$

$$\boldsymbol{B}^{-1} = \begin{bmatrix} 1/l^2 & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{bmatrix} \tag{40}$$

$$II_B = 2l + \frac{1}{l^2} \tag{41}$$

Young's Modulus and Shearing Modulus in Infinitesimal Deformation 2

- If we adopt W_R^H for W then,

$$
\begin{aligned}
\frac{\partial W_R^H}{\partial I_B} &= \frac{\partial W_R^H}{\partial \widetilde{I}_B}\frac{\partial \widetilde{I}_B}{\partial I_B} \\
&= III_B^{-\frac{1}{3}}\left\{c_1 + 2c_3\left(\widetilde{I}_B - 3\right) + c_4\left(\widetilde{II}_B - 3\right)\right. \\
&\qquad \left. +3c_6\left(\widetilde{I}_B - 3\right)^2 + 2c_7\left(\widetilde{I}_B - 3\right)\left(\widetilde{II}_B - 3\right) + c_8\left(\widetilde{II}_B - 3\right)^2\right\} \\
\frac{\partial W_R^H}{\partial II_B} &= \frac{\partial W_R^H}{\partial \widetilde{II}_B}\frac{\partial \widetilde{II}_B}{\partial II_B} \\
&= III_B^{-\frac{2}{3}}\left\{c_2 + c_4\left(\widetilde{I}_B - 3\right) + 2c_5\left(\widetilde{II}_B - 3\right)\right. \\
&\qquad \left. +c_7\left(\widetilde{I}_B - 3\right)^2 + 2c_8\left(\widetilde{I}_B - 3\right)\left(\widetilde{II}_B - 3\right) + 3c_9\left(\widetilde{II}_B - 3\right)^2\right\} \\
\frac{\partial W_R^H}{\partial III_B} &= \frac{\partial W_R^H}{\partial \widetilde{I}_B}\frac{\partial \widetilde{I}_B}{\partial III_B} + \frac{\partial W_R^H}{\partial \widetilde{II}_B}\frac{\partial \widetilde{II}_B}{\partial III_B} \\
&= -\frac{1}{3}I_B III_B^{-\frac{4}{3}}\left\{c_1 + 2c_3\left(\widetilde{I}_B - 3\right) + c_4\left(\widetilde{II}_B - 3\right)\right. \\
&\qquad \left. +3c_6\left(\widetilde{I}_B - 3\right)^2 + 2c_7\left(\widetilde{I}_B - 3\right)\left(\widetilde{II}_B - 3\right) + c_8\left(\widetilde{II}_B - 3\right)^2\right\} \\
&\qquad -\frac{2}{3}II_B III_B^{-\frac{5}{3}}\left\{c_2 + c_4\left(\widetilde{I}_B - 3\right) + 2c_5\left(\widetilde{II}_B - 3\right)\right. \\
&\qquad \left. +c_7\left(\widetilde{I}_B - 3\right)^2 + 2c_8\left(\widetilde{I}_B - 3\right)\left(\widetilde{II}_B - 3\right) + 3c_9\left(\widetilde{II}_B - 3\right)^2\right\}
\end{aligned}
$$

Young's Modulus and Shearing Modulus in Infinitesimal Deformation 2

- Cauchy stress is evaluated as,

$$T_{kl} = -p\delta_{kl} + 2\left[\left\{\frac{\partial W_R^H}{\partial II_B}(2l + \frac{1}{l}) + \frac{\partial W_R^H}{\partial III_B}\right\}\delta_{kl} + \frac{\partial W_R^H}{\partial I_B}\begin{bmatrix} l^2 & 0 & 0 \\ 0 & 1/l & 0 \\ 0 & 0 & 1/l \end{bmatrix} - \frac{\partial W_R^H}{\partial II_B}\begin{bmatrix} 1/l^2 & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{bmatrix}\right]$$

- If we pull the plane x_1, then given $T_{22} = T_{33} = 0$,

$$p = 2\left\{\frac{1}{l}\frac{\partial W_R^H}{\partial I_B} + (l + \frac{1}{l^2})\frac{\partial W_R^H}{\partial II_B} + \frac{\partial W_R^H}{\partial III_B}\right\} \quad (42)$$

$$T_{11} = 2\left\{(l^2 - \frac{1}{l})\frac{\partial W_R^H}{\partial I_B} + (l - \frac{1}{l^2})\frac{\partial W_R^H}{\partial II_B}\right\} \quad (43)$$

- Here, $l = 1 + \varepsilon$ is given under infinitesimal displacement, and having ignored ε^2, we obtain as,

$$T_{11} = 6(c_1 + c_2)\varepsilon \quad (44)$$

- $6(c1 + c2)$ is equivalent with Young's Modulus E .