Nonlinear
Finite Element Method

25/10/2004

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- Lectures include discussion of the nonlinear finite element method.
- It is preferable to have completed “Introduction to Nonlinear Finite Element Analysis” available in summer session.
- If not, students are required to study on their own before participating this course. Reference:Toshiaki.,Kubo. “Introduction: Tensor Analysis For Nonlinear Finite Element Method” (Hisennkei Yugen Yoso no tameno Tensor Kaiseki no Kiso),Maruzen.
- Lecture references are available and downloadable at http://www.sml.k.u-tokyo.ac.jp/members/nabe/lecture2004 They should be posted on the website by the day before scheduled meeting, and each students are expected to come in with a copy of the reference.
- Lecture notes from previous year are available and downloadable, also at http://www.sml.k.u.tokyo.ac.jp/members/nabe/lecture2003 You may find the course title, ”Advanced Finite Element Method” but the contents covered are the same I will cover this year.
- I will assign the exercises from this year, and expect the students to hand them in during the following lecture. They are not the requirements and they will not be graded, however it is important to actually practice calculate in deeper understanding the finite element method.
- For any questions, contact me at nabe@sml.k.u-tokyo.ac.jp

Nonlinear Finite Element Method Lecture Schedule

1. 10/ 4 Finite element analysis in boundary value problems and the differential equations
2. 10/18 Finite element analysis in linear elastic body
3. 10/25 Isoparametric solid element (program)
4. 11/ 1 Numerical solution and boundary condition processing for system of linear equations（with exercises）
5. 11/ 8 Basic program structure of the linear finite element method(program)
6. 11/15 Finite element formulation in geometric nonlinear problems(program)
7. 11/22 Static analysis technique、hyperelastic body and elastic-plastic material for nonlinear equations (program)
8. 11/29 Exercises for Lecture7
9. 12/ 6 Dynamic analysis technique and eigenvalue analysis in the nonlinear equations
10. 12/13 Structural element
11. 12/20 Numerical solution— skyline method、iterative method for the system of linear equations
12. 1/17 ALE finite element fluid analysis
13. 1/24 ALE finite element fluid analysis

Differential Equations With Boundary Value Problems

- Consider the following boundary value problems for differential equations.

[B] Find u that satisfies following condition.

$$-\frac{d^2u}{dx^2} = f(x) \qquad (0 < x < a) \tag{1}$$

$$u(0) = \alpha, \quad \frac{du}{dx}(a) = \beta \tag{2}$$

- [B] is equivalent with following [V] .

[V] Find u that satisfies the following. v is an arbitrary function with $v(0) = 0$, $v(a) = 0$

$$\int_0^a \frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^a f(x)\,v\,dx \quad \forall v \tag{3}$$

$$u(0) = \alpha, \quad \frac{du}{dx}(a) = \beta \tag{4}$$

Approximate Solutions for Weak Form— Dividing the Integration Interval

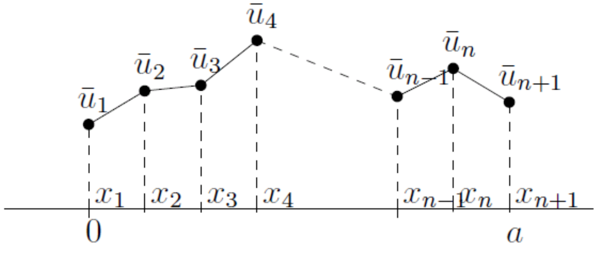
- Obtain the approximate solution for weak formulation by finite element method.
- First, divide the domain of u ,$[0, a]$ into *n* sub-intervals $[x_i, x_{i+1}]$ ($i = 1 \sim n$) without any overlaps.
- x_i ($i = 1 \sim n + 1$) is called nodal point.
- Then,

$$\int_0^a \frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^a f(x)\,v\,dx \qquad (5)$$

Reform the integration, and gain the following.

$$\sum_{i=1}^{n}\int_{x_i}^{x_{i+1}} \frac{du}{dx}\frac{dv}{dx}\,dx = \sum_{i=1}^{n}\int_{x_i}^{x_{i+1}} f\cdot v\,dx \qquad (6)$$

- Approximate solution takes the value $\bar{u}_i$ ($i = 1 \sim n + 1$) at nodal point x_i ($i = 1 \sim n + 1$) and the solution we assume to change linearly between the nodal point *xi* and *xi*+1. We use a similar function for *v.*



Finite Element Interpolation

• With the starting point x_i and the end point x_i+1, we express the integration intervals as $x_{(i,1)}, x_{(i,2)}$ and change the variables for each interval to have -1 for $x_{(i,1)}$ and 1 for $x_{(i,2)}$.

$$\sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} \frac{du}{dx}\frac{dv}{dx}\,dx = \sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} f \cdot v\,dx \tag{7}$$

• Upon using a parameter r $(-1 \leq r \leq 1)$, x in $[x_{(i,1)}, x_{(i,2)}]$ is expressed by

$$x = N^{(1)} x_{(i,1)} + N^{(2)} x_{(i,2)} \tag{8}$$

$$N^{(1)} = \frac{1}{2}(1-r), \quad N^{(2)} = \frac{1}{2}(1+r) \tag{9}$$

• $N^{(i)}$ is called the finite element method interpolation function, or simply the interpolation function.
• For $\bar{u}, \bar{v}$, we may define them to be $\bar{u}_{(i,1)} = \bar{u}_i,\ \bar{u}_{(i,2)} = \bar{u}_{i+1},\ \bar{v}_{(i,1)} = \bar{v}_i,\ \bar{v}_{(i,2)} = \bar{v}_{i+1}$ and using the parameter r $(-1 \leq r \leq 1)$ to be expressed as,

$$\bar{u} = N^{(1)} u_{(i,1)} + N^{(2)} u_{(i,2)}, \quad \bar{v} = N^{(1)} v_{(i,1)} + N^{(2)} v_{(i,2)} \tag{10}$$

Differentials in Discrete Expression

- Reformed expression for weak formulation in approximate solution.

$$\sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} \frac{d\bar{u}}{dx}\frac{d\bar{v}}{dx}\,dx = \sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} f \cdot \bar{v}\,dx \tag{11}$$

Differentials for $\bar{u}$, $\bar{v}$ about x ,which included in the left hand side integrand can be gained with using chain rule,

$$\begin{aligned} \frac{d\bar{u}}{dx} &= \frac{d}{dx}\left(N^{(1)}\bar{u}_{(i,1)} + N^{(2)}\bar{u}_{(i,2)}\right) \\ &= \frac{dN^{(1)}}{dx}\bar{u}_{(i,1)} + \frac{dN^{(2)}}{dx}\bar{u}_{(i,2)} \\ &= \frac{dN^{(1)}}{dr}\frac{dr}{dx}\bar{u}_{(i,1)} + \frac{dN^{(2)}}{dr}\frac{dr}{dx}\bar{u}_{(i,2)} \end{aligned} \tag{12}$$

- An inverse number obtained for $\frac{dr}{dx}$ appears in the equation above.

$$\begin{aligned} \frac{dx}{dr} &= \frac{d}{dx}\left(N^{(1)}x_{(i,1)} + N^{(2)}x_{(i,2)}\right) \\ &= \frac{dN^{(1)}}{dr}x_{(i,1)} + \frac{dN^{(2)}}{dr}x_{(i,2)} \end{aligned} \tag{13}$$

Element Matrix 1

- Approximation of weak formulation is expressed as,

$$\sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} \frac{d\bar{u}}{dx}\frac{d\bar{v}}{dx}\, dx = \sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} f \cdot \bar{v}\, dx \qquad (14)$$

And its left hand side integrand can be expressed in matrix form.

$$\begin{aligned} \frac{d\bar{u}}{dx}\frac{d\bar{v}}{dx} &= \left(\frac{dN^{(1)}}{dx}\bar{u}_{(i,1)} + \frac{dN^{(2)}}{dx}\bar{u}_{(i,2)}\right)\left(\frac{dN^{(1)}}{dx}\bar{v}_{(i,1)} + \frac{dN^{(2)}}{dx}\bar{v}_{(i,2)}\right) \\ &= \{\bar{v}_{(i,1)}\ \bar{v}_{(i,2)}\} \begin{bmatrix} \frac{dN^{(1)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(1)}}{dx}\frac{dN^{(2)}}{dx} \\ \frac{dN^{(2)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(2)}}{dx}\frac{dN^{(2)}}{dx} \end{bmatrix} \begin{Bmatrix} \bar{u}_{(i,1)} \\ \bar{u}_{(i,2)} \end{Bmatrix} \end{aligned} \qquad (15)$$

- With each integration interval $[x_{(i,1)}, x_{(i,2)}]$ $J_{(i)} = \frac{dx}{dr}$ then,

$$\sum_{i=1}^{n} \int_{-1}^{1} \{\bar{v}_{(i,1)}\ \bar{v}_{(i,2)}\} \begin{bmatrix} \frac{dN^{(1)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(1)}}{dx}\frac{dN^{(2)}}{dx} \\ \frac{dN^{(2)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(2)}}{dx}\frac{dN^{(2)}}{dx} \end{bmatrix} \begin{Bmatrix} \bar{u}_{(i,1)} \\ \bar{u}_{(i,2)} \end{Bmatrix} J_{(i)}\, dr = \sum_{i=1}^{n} \int_{-1}^{1} \{\bar{v}_{(i,1)}\ \bar{v}_{(i,2)}\} \begin{Bmatrix} N^{(1)} \\ N^{(2)} \end{Bmatrix} f\, J_{(i)}\, dr \qquad (16)$$

Element Matrix 2

- $\bar{v}_{(i,1)}, \bar{v}_{(i,2)}, \bar{u}_{(i,1)}, \bar{u}_{(i,2)}$ represent values at nodal points, and since integration variables *r* are constant numbers, we can pull them out of the integrations.

$$\sum_{i=1}^{n} \int_{-1}^{1} \{\bar{v}_{(i,1)}\ \bar{v}_{(i,2)}\} \begin{bmatrix} \frac{dN^{(1)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(1)}}{dx}\frac{dN^{(2)}}{dx} \\ \frac{dN^{(2)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(2)}}{dx}\frac{dN^{(2)}}{dx} \end{bmatrix} \begin{Bmatrix} \bar{u}_{(i,1)} \\ \bar{u}_{(i,2)} \end{Bmatrix} J_{(i)}\, dr$$
$$= \sum_{i=1}^{n} \{\bar{v}_{(i,1)}\ \bar{v}_{(i,2)}\} \int_{-1}^{1} \begin{bmatrix} \frac{dN^{(1)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(1)}}{dx}\frac{dN^{(2)}}{dx} \\ \frac{dN^{(2)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(2)}}{dx}\frac{dN^{(2)}}{dx} \end{bmatrix} J_{(i)}\, dr \begin{Bmatrix} \bar{u}_{(i,1)} \\ \bar{u}_{(i,2)} \end{Bmatrix} \tag{17}$$

$$\sum_{i=1}^{n} \int_{-1}^{1} \{\bar{v}_{(i,1)}\ \bar{v}_{(i,2)}\} \begin{Bmatrix} N^{(1)} \\ N^{(2)} \end{Bmatrix} f\, J_{(i)}\, dr = \sum_{i=1}^{n} \{\bar{v}_{(i,1)}\ \bar{v}_{(i,2)}\} \int_{-1}^{1} \begin{Bmatrix} N^{(1)} \\ N^{(2)} \end{Bmatrix} f\, J_{(i)}\, dr \tag{18}$$

- $[K^{(i)}]$, which defines $[K^{(i)}]$, $\{F^{(i)}\}$ is called element matrix.

$$[K^{(i)}] = \begin{bmatrix} K_{11}^{(i)} & K_{12}^{(i)} \\ K_{21}^{(i)} & K_{22}^{(i)} \end{bmatrix} = \int_{-1}^{1} \begin{bmatrix} \frac{dN^{(1)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(1)}}{dx}\frac{dN^{(2)}}{dx} \\ \frac{dN^{(1)}}{dx}\frac{dN^{(2)}}{dx} & \frac{dN^{(2)}}{dx}\frac{dN^{(2)}}{dx} \end{bmatrix} J_{(i)}\, dr \quad \{F^{(i)}\} = \begin{Bmatrix} F_1^{(i)} \\ F_2^{(i)} \end{Bmatrix} = \int_{-1}^{1} \begin{Bmatrix} N^{(1)} \\ N^{(2)} \end{Bmatrix} f\, J_{(i)}\, dr \tag{19}$$

High Order Interpolation

- Approximate the function u by quadratic function.
- Divide the domain $[0, a]$ for u into n subdivisions of intervals $Ii(i = 1 \sim n)$. We need three points in order to define the quadratic function thus, we take $2n + 1$ nodal points.

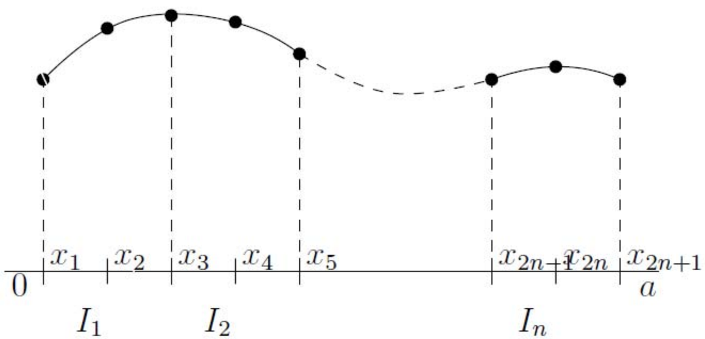


図 1: 2 次関数による補間

- The nodal points included in the interval I_i: x_{2i-1}, x_{2i}, x_{2i+1} ,are expressed by $x_{(i,1)}$, $x_{(i,3)}$, $x_{(i,2)}$ so, we can obtain the interval $I_i = [x_{(i,1)}, x_{(i,2)}]$

$$\sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} \frac{du}{dx}\frac{dv}{dx}\,dx = \sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} f \cdot v\,dx \qquad (20)$$

High Order Finite Element Interpolation 1

- In each integration interval, conduct coordinates transform for $x(i,1)$ to be -1 and $x(i,2)$ to be 1. Using a parameter r $(-1 \le r \le 1)$, $x \in [x(i,1), x(i,2)]$ can be expressed by,

$$x = N^{(1)} x_{(i,1)} + N^{(2)} x_{(i,2)} + N^{(3)} x_{(i,3)} \tag{21}$$

Only if

$$N^{(1)} = -\frac{1}{2} r (1-r), \quad N^{(2)} = \frac{1}{2} r (1+r), \quad N^{(3)} = 1 - r^2 \tag{22}$$

- Do the same for u with parameter r $(-1 \le r \le 1)$, with conditions provided by $u_{(i,1)} = u_{2i-1}$ $u_{(i,2)} = u_{2i+1}$, $u_{(i,3)} = u_{2i}$.

$$u = N^{(1)} u_{(i,1)} + N^{(2)} u_{(i,2)} + N^{(3)} u_{(i,3)} \tag{23}$$

- Using chain rule, following is obtained.

$$\frac{du}{dx} = \frac{dN^{(1)}}{dx} u_{(i,1)} + \frac{dN^{(2)}}{dx} u_{(i,2)} + \frac{dN^{(3)}}{dx} u_{(i,3)} = \frac{dN^{(1)}}{dr}\frac{dr}{dx} u_{(i,1)} + \frac{dN^{(2)}}{dr}\frac{dr}{dx} u_{(i,2)} + \frac{dN^{(3)}}{dr}\frac{dr}{dx} u_{(i,3)} \tag{24}$$

- $\dfrac{dr}{dx}$ may be obtained as an inverse.

$$\frac{dx}{dr} = \frac{dN^{(1)}}{dr} x_{(i,1)} + \frac{dN^{(2)}}{dr} x_{(i,2)} + \frac{dN^{(3)}}{dr} x_{(i,3)} \tag{25}$$

High Order Finite Element Interpolation 2

• We have obtained , $x_{(i,3)} = (x_{(i,1)} + x_{(i,2)})/2$. Based on this we have,

$$\begin{aligned} x &= N^{(1)}\, x_{(i,1)} + N^{(2)}\, x_{(i,2)} + N^{(3)}\, x_{(i,3)} \\ &= \left\{ -\frac{1}{2} r\,(1-r) \right\} x_{(i,1)} + \left\{ \frac{1}{2} r\,(1+r) \right\} x_{(i,2)} + \left(1-r^2\right) \frac{1}{2} \left(x_{(i,1)} + x_{(i,2)}\right) \\ &= \left\{ \frac{1}{2}\,(1-r) \right\} x_{(i,1)} + \left\{ \frac{1}{2}\,(1+r) \right\} x_{(i,2)} \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{dx}{dr} &= \frac{dN^{(1)}}{dr} x_{(i,1)} + \frac{dN^{(2)}}{dr} x_{(i,2)} + \frac{dN^{(3)}}{dr} x_{(i,3)} \\ &= \left\{ -\frac{1}{2}\,(1-2r) \right\} x_{(i,1)} + \left\{ \frac{1}{2}\,(1+2r) \right\} x_{(i,2)} + (-2r)\, \frac{1}{2} \left(x_{(i,1)} + x_{(i,2)}\right) \\ &= -\frac{1}{2} x_{(i,1)} + \frac{1}{2} x_{(i,2)} \end{aligned} \tag{27}$$

• The above represents the interpolation function under linearly changing x within the interval, and which basically implies that the same results are obtained by the two points interpolations.
• For those coordinates interpolation function with lower order than the displacement, are called sumparametric element, while those with opposite condition are called super parametric element.
• Anyhow, most commonly, isoparametric elements are being used in practical sense.

Element Matrix

- If we take $J_{(i)} = \left|\dfrac{dx}{dr}\right|$ to correspond with each integration interval $[x_{(i,1)}, x_{(i,2)}]$,

$$\sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} \frac{du}{dx}\frac{dv}{dx}\, dx = \sum_{i=1}^{n} \int_{x_{(i,1)}}^{x_{(i,2)}} f \cdot v\, dx \tag{28}$$

$\Leftrightarrow$

$$\sum_{i=1}^{n} \int_{-1}^{1} \{v_{(i,1)}\, v_{(i,2)}\, v_{(i,3)}\} \begin{bmatrix} \frac{dN^{(1)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(1)}}{dx}\frac{dN^{(2)}}{dx} & \frac{dN^{(1)}}{dx}\frac{dN^{(3)}}{dx} \\ \frac{dN^{(2)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(2)}}{dx}\frac{dN^{(2)}}{dx} & \frac{dN^{(2)}}{dx}\frac{dN^{(3)}}{dx} \\ \frac{dN^{(3)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(3)}}{dx}\frac{dN^{(2)}}{dx} & \frac{dN^{(3)}}{dx}\frac{dN^{(3)}}{dx} \end{bmatrix} \begin{Bmatrix} u_{(i,1)} \\ u_{(i,2)} \\ u_{(i,3)} \end{Bmatrix} J_{(i)}\, dr = \sum_{i=1}^{n} \int_{-1}^{1} \{v_{(i,1)}\, v_{(i,2)}\, v_{(i,3)}\} \begin{Bmatrix} N^{(1)} \\ N^{(2)} \\ N^{(3)} \end{Bmatrix} f\, J_{(i)}\, dr \tag{29}$$

- $v_{(i,1)}, v_{(i,2)}, v_{(i,3)}, u_{(i,1)}, u_{(i,2)}, u_{(i,3)}$, represent the values at nodal points, and they stay constant about r , thus they can be pulled out of the integrals. The following equations define $[K^{(i)}]$, $\{F^{(i)}\}$.

$$[K^{(i)}] = \begin{bmatrix} K_{11}^{(i)} & K_{12}^{(i)} & K_{13}^{(i)} \\ K_{21}^{(i)} & K_{22}^{(i)} & K_{23}^{(i)} \\ K_{31}^{(i)} & K_{32}^{(i)} & K_{33}^{(i)} \end{bmatrix} = \int_{-1}^{1} \begin{bmatrix} \frac{dN^{(1)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(1)}}{dx}\frac{dN^{(2)}}{dx} & \frac{dN^{(1)}}{dx}\frac{dN^{(3)}}{dx} \\ \frac{dN^{(2)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(2)}}{dx}\frac{dN^{(2)}}{dx} & \frac{dN^{(2)}}{dx}\frac{dN^{(3)}}{dx} \\ \frac{dN^{(3)}}{dx}\frac{dN^{(1)}}{dx} & \frac{dN^{(3)}}{dx}\frac{dN^{(2)}}{dx} & \frac{dN^{(3)}}{dx}\frac{dN^{(3)}}{dx} \end{bmatrix} J_{(i)}\, dr \tag{30}$$

$$\{F^{(i)}\} = \begin{Bmatrix} F_1^{(i)} \\ F_2^{(i)} \\ F_3^{(i)} \end{Bmatrix} = \int_{-1}^{1} \begin{Bmatrix} N^{(1)} \\ N^{(2)} \\ N^{(3)} \end{Bmatrix} f\, J_{(i)}\, dr \tag{31}$$

Numerical Integration

- It is necessary to conduct either volume or area integration in obtaining the matrix.
- However, it is almost impossible to analytically conduct integration because the integrand becomes complicated.
- Thus we conduct numerical integration instead, and Newton-Coate integration along with Gauss integrations are among the most common methods.
- Both integrations approximate the integrand by Lagrange polynomials based on the characteristics of Lagrange polynomials to obtain integration numerically.

Lagrange Polynomials 1

- Approximate $f(x)$, $(a \le x \le b)$ by polynomials.
- Lagrange polynomials take the sampling points including both extremes of domain: $\{x_n\}, (a = x_1 < x_2 < \cdots < x_n = b)$ to be approximated by following,

$$f(x) \approx Q_n(x) = \sum_{k=1}^{n} f(x_k) H_k(x) \tag{32}$$

$$H_i(x) = \frac{(x - x_1)(x - x_2) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n)}{(x_i - x_1)(x_i - x_2) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)} \tag{33}$$

- H_k is $n-1$th order function that takes 1 at the sampling points, and 0 at any other points.

$$H_k(x_i) = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases} \tag{34}$$

- Thus the sampling points x_k,

$$f(x_k) = Q_n(x_k) \tag{35}$$

$Q_n(x)$ is $n-1$th order function, which coincides with $f(x)$ with n sampling points $i(i = 1, \cdots, n)$.

- For example, when $n = 2$ we have $x_1 = a, x_2 = b$

$$H_1(x) = \frac{x - b}{a - b}, \quad H_2(x) = \frac{x - a}{b - a} \tag{36}$$

$$f(x) \approx Q_n(x) = \sum_{k=1}^{n} f(x_k) H_k(x) = f(a)\frac{x - b}{a - b} + f(b)\frac{x - a}{b - a} \tag{37}$$

This represents a straight line connected by the end points.

- If we take $x_1 = -1, x_2 = 1$,

$$H_1(x) = \frac{1}{2}(1 - x), \quad H_2(x) = \frac{1}{2}(1 + x) \tag{38}$$

Then we obtain the above, which coincide with the previous interpolation function in the single order.

Lagrange Polynomials 2

- Basic facts：When two *n*th- order polynomials $f(x)$, $g(x)$ coincide with another $n+1$ points $x_i (i = 1, \cdots, n+1)$, then f coincide with g, as well.

  Proof：Suppose we have $h(x) = f(x) - g(x)$ then $h(x)$ takes n th order polynomials. Now, under $x_i (i = 1, \cdots, n+1)$, if $f(x)$ coincides with $g(x)$,

$$\begin{aligned} f(x_i) &= g(x_i) \quad (i = 1, \cdots, n+1) \\ h(x_i) &= f(x_i) - g(x_i) = 0 \quad (i = 1, \cdots, n+1) \\ &= a(x - x_1)(x - x_2) \cdots (x - x_{n+1}) = 0 \end{aligned} \tag{39}$$

  Where *a* is an arbitrary coefficient. Hence, $h(x)$ becomes $n+1$ th order function and there appears a contradiction.

- If we take $f(x)$ as n th order polynomials to approximate by Lagrange polynomials. For each $H_k(x)$ n th order function is taken with n+1 sampling points, $Q_{n+1}(x)$ becomes n *th* order function. Based on the facts, $f(x)$ and $Q_{n+1}(x)$ coincide regardless of how the sampling points are taken.

Approximation of Low-level Function by Lagrange Polynomials 1

- We need two points when $f(x)$ is the first order function.

$$f(x) = Q_2(x) = f(x_1)H_1^{(2)}(x) + f(x_2)H_2^{(2)}(x) \tag{40}$$

- Consider now for approximation of first order function with three points.

$$f(x) \approx Q_3(x) = f(x_1)H_1^{(3)}(x) + f(x_2)H_2^{(3)}(x) + f(x_3)H_3^{(3)}(x) \tag{41}$$

Using the end points x_1, x_3 ,

Be careful that sampling points differ from usual Lagrane polynomials.

$$f(x) = Q_2(x) = f(x_1)H_1^{(2)}(x) + f(x_3)H_3^{(2)}(x) \tag{42}$$

For x_2 ,

$$H_1^{(2)}(x) = \frac{x - x_3}{x_1 - x_3}, \quad H_3^{(2)}(x) = \frac{x - x_1}{x_3 - x_1} \tag{43}$$

Substitute the above into $f(x) \approx Q_3$

$$f(x_2) = Q_2(x_2) = f(x_1)H_1^{(2)}(x_2) + f(x_3)H_3^{(2)}(x_2) \tag{44}$$

$$\begin{aligned} f(x) \approx Q_3(x) &= f(x_1)H_1^{(3)}(x) + Q_2(x_2)H_2^{(3)}(x) + f(x_3)H_3^{(3)}(x) \\ &= f(x_1)H_1^{(3)}(x) + \{f(x_1)H_1^{(2)}(x_2) + f(x_3)H_3^{(2)}(x_2)\}H_2^{(3)}(x) + f(x_3)H_3^{(3)}(x) \\ &= f(x_1)\{H_1^{(3)}(x) + H_1^{(2)}(x_2)H_2^{(3)}(x)\} \\ &+ f(x_3)\{H_3^{(3)}(x) + H_3^{(2)}(x_2)H_2^{(3)}(x)\} \end{aligned} \tag{45}$$

$$\begin{aligned} H_1^{(3)}(x) + H_1^{(2)}(x_2)H_2^{(3)}(x) &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + \frac{x_2-x_3}{x_1-x_3}\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \\ = \frac{x-x_3}{(x_1-x_2)(x_1-x_3)}\{(x-x_2)-(x-x_1)\} &= \frac{(x-x_3)(x_1-x_2)}{(x_1-x_2)(x_1-x_3)} = \frac{x-x_3}{x_1-x_3} \end{aligned} \tag{46}$$

$$\begin{aligned} H_3^{(3)}(x) + H_3^{(2)}(x_2)H_2^{(3)}(x) &= \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} + \frac{x_2-x_1}{x_3-x_1}\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \\ = \frac{x-x_1}{(x_3-x_1)(x_3-x_3)}\{(x-x_2)-(x-x_3)\} &= \frac{(x-x_1)(x_2-x_3)}{(x_3-x_1)(x_3-x_2)} = \frac{x-x_1}{x_3-x_1} \end{aligned} \tag{47}$$

From above, obtained by following.

$$\begin{aligned} f(x) \approx Q_3(x) &= f(x_1)\frac{x - x_3}{x_1 - x_3} + f(x_3)\frac{x - x_1}{x_3 - x_1} \\ &= f(x_1)H_1^{(2)}(x) + f(x_3)H_3^{(2)}(x) \end{aligned} \tag{48}$$

When approximate $f(x)$ by $Q_3(x)$, $Q_3(x)$ becomes the first order function thus coincides with the approximation by $Q_2(x)$.

Approximation of Low-level Function by Lagrange Polynomials 2

• Three points are needed for quadratic $f(x)$.

$$f(x) = Q_3(x) = f(x_1)H_1^{(3)}(x) + f(x_2)H_2^{(3)}(x) + f(x_3)H_3^{(3)}(x) \tag{49}$$

•Approximation by 4 points,

$$f(x) \approx Q_4(x) = f(x_1)H_1^{(4)}(x) + f(x_2)H_2^{(4)}(x) + f(x_3)H_3^{(4)}(x) + f(x_4)H_4^{(4)}(x) \tag{50}$$

Pick the three points other than x_2 (same argument for x_3)

$$f(x) = Q_3(x) = f(x_1)H_1^{(3)}(x) + f(x_3)H_3^{(3)}(x) + f(x_4)H_4^{(3)}(x) \tag{51}$$

When x_2

$$f(x_2) = Q_3(x_2) = f(x_1)H_1^{(3)}(x_2) + f(x_3)H_3^{(3)}(x_2) + f(x_4)H_4^{(3)}(x_2) \tag{52}$$

Substitute above into $f(x) \approx Q_4(x)$

$$\begin{aligned}
f(x) \approx Q_4(x) &= f(x_1)H_1^{(4)}(x) + Q_3(x_2)H_2^{(4)}(x) + f(x_3)H_3^{(4)}(x) + f(x_4)H_4^{(4)}(x) \\
&= f(x_1)H_1^{(4)}(x) \\
&+ \{f(x_1)H_1^{(3)}(x_2) + f(x_3)H_3^{(3)}(x_2) + f(x_4)H_4^{(3)}(x_2)\} \\
&+ H_2^{(4)}(x) + f(x_3)H_3^{(4)}(x) + f(x_4)H_4^{(4)}(x) \\
&= f(x_1)\{H_1^{(4)}(x) + H_1^{(3)}(x_2)H_2^{(4)}(x)\} \\
&+ f(x_3)\{H_3^{(4)}(x) + H_3^{(3)}(x_2)H_2^{(4)}(x)\} \\
&+ f(x_4)\{H_4^{(4)}(x) + H_4^{(3)}(x_2)H_2^{(4)}(x)\}
\end{aligned} \tag{53}$$

$$\begin{aligned}
&H_1^{(4)}(x) + H_1^{(3)}(x_2)H_2^{(4)}(x) \\
&= \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{(x_2-x_3)(x_2-x_4)}{(x_1-x_3)(x_1-x_4)} \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
&= \frac{(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} \{(x-x_2)-(x-x_1)\} = \frac{(x-x_3)(x-x_4)}{(x_1-x_3)(x_1-x_4)}
\end{aligned} \tag{54}$$

$$\begin{aligned}
&H_3^{(4)}(x) + H_3^{(3)}(x_2)H_2^{(4)}(x) \\
&= \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{(x_2-x_1)(x_2-x_4)}{(x_3-x_1)(x_3-x_4)} \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
&= \frac{(x-x_1)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_1-x_4)} \{(x-x_2)-(x-x_3)\} = \frac{(x-x_1)(x-x_4)}{(x_3-x_1)(x_3-x_4)}
\end{aligned} \tag{55}$$

$$
\begin{aligned}
&H_4^{(4)}(x) + H_4^{(3)}(x_2)H_2^{(4)}(x) \\
&= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} + \frac{(x_2-x_1)(x_2-x_3)}{(x_4-x_1)(x_4-x_3)} \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\
&= \frac{(x-x_1)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \{(x-x_2)-(x-x_4)\} = \frac{(x-x_1)(x-x_3)}{(x_4-x_1)(x_4-x_3)}
\end{aligned}
\tag{56}
$$

So,

$$
\begin{aligned}
f(x) \approx Q_4(x) = {} & f(x_1)\frac{(x-x_3)(x-x_4)}{(x_1-x_3)(x_1-x_4)} \\
& + f(x_3)\frac{(x-x_1)(x-x_4)}{(x_3-x_1)(x_3-x_4)} \\
& + f(x_4)\frac{(x-x_1)(x-x_3)}{(x_4-x_1)(x_4-x_3)} \\
& = f(x_1)H_1^{(3)}(x) + f(x_3)H_3^{(3)}(x) + f(x_4)H_4^{(3)}(x)
\end{aligned}
\tag{57}
$$

Approximation by $Q_4(x)$ becomes quadratic thus coincide with the approximation by $Q_3(x)$.

Basics to Numerical Integration

- Newton-Coate integration and Gauss integration are the method of numerically obtaining the integration based on the appro

$$\int_a^b f(x)dx \approx \int_a^b Q_n(x)dx = \int_a^b \sum_{k=1}^{n} f(x_k)H_k(x)dx = \sum_{k=1}^{n} f(x_k)\int_a^b H_k(x)dx \quad (58)$$

- The following integration value is gained regardless of $f(x)$, but rather gained based on the information of the sampling points, and which is called the heaviness corresponding to the sampling points xk.

$$w_k = \int_a^b H_k(x)dx \quad (59)$$

- Therefore, we can obtain the approximation by multiplying the heaviness, corresponding to the value at sampling points $f(xk)$ and the point xk, to the integration of $f(x)$ then add them all together.

$$\int_a^b f(x)dx \approx \int_a^b Q_n(x)dx = \sum_{k=1}^{n} f(x_k)w_k \quad (60)$$

- Since integrand is approximated by Lagrange polynomials, we gain more accuracy with greater the number of the sampling points.

- However when integrand is *n th* polynomials, the solution coincides with analytical integration by taking the $n + 1$sampling points. And we observe no difference by taking more than $n + 2$ sampling points.
- $x = \frac{a}{2}(1-r) + \frac{b}{2}(1+r)$ then,

$$\int_a^b f(x)dx = \int_{-1}^{1} f(x(r))\frac{dx}{dr}dr = \int_{-1}^{1} f(x(r))\frac{b-a}{2}dr = \frac{b-a}{2}\int_{-1}^{1} f(x(r))dr \quad (61)$$

Thus,

$$\begin{aligned}
\int_a^b f(x)dx &= \frac{b-a}{2}\int_{-1}^{1} f(x(r))dr \approx \frac{b-a}{2}\int_{-1}^{1} Q_n(r)dr \\
&= \frac{b-a}{2}\int_{-1}^{1}\sum_{k=1}^{n} f(x_k(r))H_k(r)dr = \frac{b-a}{2}\sum_{k=1}^{n} f(x_k(r))\int_{-1}^{1} H_k(r)dx \qquad (62)
\end{aligned}$$

Discussion follows with a set integration interval from -1 to 1.

Newton-Coate Integration 1

- In Newton-Coate integration, the end points are included in selecting equal intervals of n sampling points.
- For $n = 2$ it is commonly called the trapezoidal rule, while $n = 3$, it is called Simpson integration.
- For the Trapezoidal rule,

$$H_1(x) = \frac{1}{2}(1 - x), \quad H_2(x) = \frac{1}{2}(1 + x) \tag{63}$$

Therefore obtained by following,

$$\begin{aligned} w_1 &= \int_{-1}^{1} H_1(x)dx = \frac{1}{2}\int_{-1}^{1}(1 - x)dx = 1 \\ w_2 &= \int_{-1}^{1} H_2(x)dx = \frac{1}{2}\int_{-1}^{1}(1 + x)dx = 1 \end{aligned} \tag{64}$$

Newton-Coate Integratoin 2

• In Simpson integration,

$$H_1(x) = \frac{1}{2}x(x-1)$$
$$H_2(x) = 1 - x^2$$
$$H_3(x) = \frac{1}{2}x(x+1) \qquad (65)$$

Therefore obtained by following

$$w_1 = \int_{-1}^{1} H_1(x)dx = \int_{-1}^{1} \frac{1}{2}x(x-1)dx = \frac{1}{3}$$
$$w_2 = \int_{-1}^{1} H_2(x)dx = \int_{-1}^{1} 1 - x^2 dx = \frac{4}{3}$$
$$w_3 = \int_{-1}^{1} H_3(x)dx = \int_{-1}^{1} \frac{1}{2}x(x+1)dx = \frac{1}{3} \qquad (66)$$

• Obviously, when we obtain the integrand with $n-1$ polynomials, the integrals can be achieved by taking more than n sampling points.
• In considering the odd function to have its integrals 0, $(2n-1)$ polynomials can be accurately obtained if $(2n-1)$ sampling points are taken.
• Thus in conducting Newton-Coate integration, often odd numbers of sampling points are taken.

Gauss Integration 1

- In Gauss integration, integrand is approximated by $(2n-1)$ order function.

$$f(x) \approx R_n(x) = \underbrace{\sum_{k=1}^{n} f(x_k) H_k(x)}_{Q_n(x)} + q(x) \sum_{k=1}^{n} a_k x^{k-1} \tag{67}$$

a_k takes an arbitrary coefficient, and $q(x)$ expresses the following n polynomials.

$$q(x) = (x - x_1)(x - x_2) \cdots (x - x_n) \tag{68}$$

- At sampling point x_k: $Q_n(x_k) = f(k_k,)$, $q(x_k) = 0$ thus,

$$f(x_k) = R_n(x_k) \tag{69}$$

- Here the position of sampling points $x_k (k = 1, \cdots, n)$ is expressed by

$$\int_{-1}^{1} q(x) x^{k-1} dx = 0 \tag{70}$$

In order to satisfy the above,

$$
\begin{aligned}
\int_{-1}^{1} f(x)dx &\approx \int_{-1}^{1} R_n(x)dx \\
&= \int_{-1}^{1} \sum_{k=1}^{n} f(x_k)H_k(x)dx + \int_{-1}^{1} q(x) \sum_{k=1}^{n} a_k x^{k-1} dx \\
&= \int_{-1}^{1} \sum_{k=1}^{n} f(x_k)H_k(x)dx + \sum_{k=1}^{n} a_k \underbrace{\int_{-1}^{1} q(x)x^{k-1}dx}_{0} \\
&= \sum_{k=1}^{n} f(x_k) \int_{-1}^{1} H_k(x)dx \\
&= \sum_{k=1}^{n} w_k f(x_k) \qquad (71)
\end{aligned}
$$

- Implying that integral of the integrand $f(x)$ is approximated as an integral of $2n-1$ th order function at n sampling points.

Gauss Integration 2

- Let us now find the specific positions for sampling points.
- When $n = 1$

$$\int_{-1}^{1} (x - x_1)x^{1-1}dx = \int_{-1}^{1} -x_1 dx$$
$$= 2x_1 = 0 \tag{72}$$

From which to obtain the heaviness that corresponds to $x_1 = 0$,

$$h_1(x) = 1$$
$$\int_{-1}^{1} h_1(x)dx = \int_{-1}^{1} 1dx = 2 \tag{73}$$

- When $n = 2$,

$$\int_{-1}^{1} (x - x_1)(x - x_2)x^{1-1}dx = \frac{2}{3} + 2x_1x_2 = 0$$
$$\int_{-1}^{1} (x - x_1)(x - x_2)x^{2-1}dx = -\frac{2}{3}(x_1 + x_2) = 0 \tag{74}$$

Then obtain heaviness corresponding to $x_1 = -\sqrt{\frac{1}{3}}, x_2 = \sqrt{\frac{1}{3}}$,

$$h_1(x) = \frac{x - \sqrt{\frac{1}{3}}}{-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}} = -\frac{\sqrt{3}}{2}\left(x - \sqrt{\frac{1}{3}}\right), \quad \int_{-1}^{1} h_1(x)dx = 1$$

$$h_2(x) = \frac{x + \sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}} = \frac{\sqrt{3}}{2}\left(x + \sqrt{\frac{1}{3}}\right), \quad \int_{-1}^{1} h_2(x)dx = 1 \qquad (75)$$

Gauss Integration 3

• When $n = 3$,

$$\int_{-1}^{1}(x-x_1)(x-x_2)(x-x_3)x^{1-1}dx$$
$$= -\frac{2}{3}(x_1+x_2+x_3) - 2(x_1x_2x_3) = 0$$
$$\int_{-1}^{1}(x-x_1)(x-x_2)(x-x_3)x^{2-1}dx$$
$$= \frac{2}{5} + \frac{2}{3}(x_1x_2+x_2x_3+x_3x_1) = 0$$
$$\int_{-1}^{1}(x-x_1)(x-x_2)(x-x_3)x^{3-1}dx$$
$$= -\frac{2}{5}(x_1+x_2+x_3) - \frac{2}{3}(x_1x_2x_3) = 0 \qquad (76)$$

Find the heaviness that correlates with $x_1 = -\sqrt{\frac{3}{5}}, x_2 = 0, x_3 = \sqrt{\frac{3}{5}}$,

$$
\begin{aligned}
h_1(x) &= \frac{x\left(x-\sqrt{\frac{3}{5}}\right)}{-\sqrt{\frac{3}{5}}2\sqrt{\frac{3}{5}}}, \quad \int_{-1}^{1} h_1(x)dx = \frac{5}{9} \\
h_2(x) &= \frac{\left(x-\sqrt{\frac{3}{5}}\right)\left(x+\sqrt{\frac{3}{5}}\right)}{-\sqrt{\frac{3}{5}}\sqrt{\frac{3}{5}}}, \quad \int_{-1}^{1} h_2(x)dx = \frac{8}{9} \\
h_3(x) &= \frac{x\left(x+\sqrt{\frac{3}{5}}\right)}{\sqrt{\frac{3}{5}}2\sqrt{\frac{3}{5}}}, \quad \int_{-1}^{1} h_3(x)dx = \frac{5}{9}
\end{aligned}
\tag{77}
$$

Sampling Points in Actual Numerical Integration

- Obviously，the more we have the sampling points, the more accurate the solution we obtain.
- However，the more we have the sampling points, greater the amount of time spent on the calculation.
- Usually, in the first-order element, 2points taken by Gauss integration and 3points by Newton-Coate integration. In the second-order element，3points used in Gauss integration and 5points used in Newton-Coate integration.

サンプリング点数	x_i	w_i
1	0	2
2	±0.57735 02691 89626	1
3	±0.77459 66692 41483	0.55555 55555 55556
	0	0.88888 88888 88889
4	±0.86113 63115 94053	0.34785 48451 37454
	±0.33998 10435 84856	0.65214 51548 62546
5	±0.90617 98459 38664	0.23692 68850 56189
	±0.53846 93101 05683	0.47862 86704 99366
	0	0.56888 88888 88889
6	±0.93246 95142 03152	0.17132 44923 79170
	±0.66120 93864 66265	0.36076 15730 48139
	±0.23861 91860 83197	0.46791 39345 72691

表 1: Gauss 積分のサンプリング点と重みの値

Proper Use for Newton-Coate and Gauss Integration

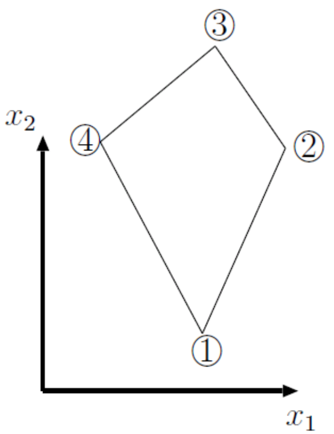
- Generally, Gauss integration should be used in the case where we have the same number of integration points because the integration possesses more accuracy.
- However,Gauss integration may be applied only when we have simple shapes such as rectangular.
- For a difficult cross section including a cylinder(even though its shape is not complicated), there is no choice but to use Newton-Coate integration (because in this case, Gauss integration does not work out).
- In finite element method, the interior element information is estimated only by the integration points. In elastic-plastic body, for example, the undulation in the material occurs within the surface, yet Gauss integration will not be able to deal with the integration points on surface, thus an accurate estimation cannot be carried out. In such cases, Newton-Coate integration is considered to be more suitable since the integration contains the integration points on surface.

4 Noded Quadrilateral Solid Element

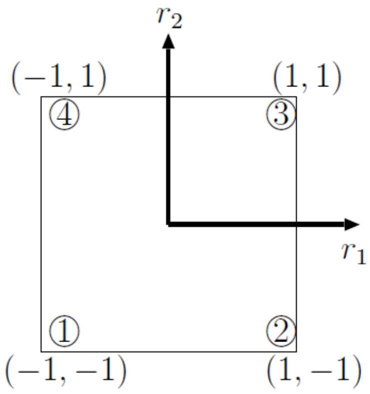
- In one-dimensional space, divide the domain of integration into *n*-interval then conduct the coordinate transformation at each interval of x coordinates in a linear segment of line to $r(-1 \leq r \leq 1)$, using interpolation function.

$$N^{(1)} = \frac{1}{2}(1-r), \quad N^{(2)} = \frac{1}{2}(1+r) \tag{78}$$

- Now, what do we find under two-dimensional space?
- First, divide domain of integration by rectangular with its apexes at $(-1,-1)$, $(1,-1)$, $(1, 1)$,and $(-1, 1)$, then conduct coordinate transformation using two parameters $r_1, r_2\ (-1 \leq r_1 \leq 1, -1 \leq r_2 \leq 1)$.
- Therefore, in physical coordinate systems, the nodal points under such configuration in the figure on the left is made to correlate with what it shows in the figures on the right. This implies that a tetrahedron in the physical coordinate system is being projected to a square in r_1- r_2 coordinate system.



(a) 物理座標系



(b) 自然座標系

Interpolation Function

- Interpolation functions takes forms in the following,

$$N^{(1)} = \frac{1}{4}(1 - r_1)(1 - r_2) \tag{79}$$

$$N^{(2)} = \frac{1}{4}(1 + r_1)(1 - r_2) \tag{80}$$

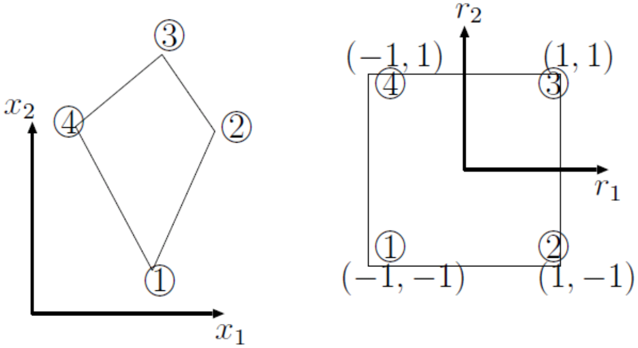
$$N^{(3)} = \frac{1}{4}(1 + r_1)(1 + r_2) \tag{81}$$

$$N^{(4)} = \frac{1}{4}(1 - r_1)(1 + r_2) \tag{82}$$

- In respect with one-dimensional space,

$$N^{(1)} = \frac{1}{2}(1 - r), \quad N^{(2)} = \frac{1}{2}(1 + r) \tag{83}$$

- Values at corresponding nodal points are found as 1, but in other nodal points, found as 0.



(a) 物理座標系 (b) 自然座標系

Differentials in Discrete Expression 1

- Differentials of ui about xj, which are needed in calculating a strain, can be evaluated with chain rule in the following.

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial N^{(n)}}{\partial x_j} u_i^{(n)} = \left(\frac{\partial N^{(n)}}{\partial r_1} \frac{\partial r_1}{\partial x_j} + \frac{\partial N^{(n)}}{\partial r_2} \frac{\partial r_2}{\partial x_j} \right) u_i^{(n)} \tag{84}$$

- $\frac{\partial N^{(n)}}{\partial x_j}$ can be obtained also, with chain rule.
- Jacobian matrix $[J]$ may be found as,

$$\begin{bmatrix} \frac{\partial N^{(n)}}{\partial r_1} \\ \frac{\partial N^{(n)}}{\partial r_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_2}{\partial r_1} \\ \frac{\partial x_1}{\partial r_2} & \frac{\partial x_2}{\partial r_2} \end{bmatrix} \begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} \tag{85}$$

$$= [J] \begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} \tag{86}$$

Differentials in Discrete Expression 2

- Each component of this Jacobian matrix $\frac{\partial x_i}{\partial r_j}$ is given by,

$$\frac{\partial x_i}{\partial r_j} = \frac{\partial N^{(n)}}{\partial r_j} x_i^{(n)} \tag{87}$$

- $\frac{\partial N^{(n)}}{\partial x_i}$ is evaluated as,

$$\begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial N^{(n)}}{\partial r_1} \\ \frac{\partial N^{(n)}}{\partial r_2} \end{bmatrix} \tag{88}$$

- In addition, the regional integration can be expressed by,

$$\int_{\Omega_e} d\Omega = \int_{-1}^{1} \int_{-1}^{1} \det[J] dr_1\, dr_2 \tag{89}$$

- This integration is usually conducted by numerical integration method such as Gauss integration. Here, we use a doubled Gauss integration in one-dimensional space.

$$\int_{-1}^{1} \int_{-1}^{1} f(x, y)\, dx\, dy \approx \sum_i \sum_j w_i w_j f(x_i, y_j) \tag{90}$$

8 Noded hexahedron Solid Element

- In one-dimensional space, divide the domain of integration into n-interval then conduct the coordinate transformation at each interval of x coordinates in a linear segment of line to $r(-1 \leq r \leq 1)$, using interpolation function.

$$N^{(1)} = \frac{1}{2}(1-r), \quad N^{(2)} = \frac{1}{2}(1+r) \tag{91}$$

- What do we find in three-dimensional space?
- First, divide the domain of integration by hexahedron with its apexes shown in the chart below, and conduct coordinate transformation for three domains using parameters r_1, r_2, and r_3 $(-1 \leq r_1 \leq 1, -1 \leq r_2 \leq 1, -1 \leq r_3 \leq 1)$

節点	r_1	r_2	r_3
1	-1	-1	-1
2	1	-1	-1
3	1	1	-1
4	-1	1	-1
5	-1	-1	1
6	1	-1	1
7	1	1	1
8	-1	1	1

表 2: 節点の対応

Physical Coordinates System and Natural Coordinates System

- In the physical coordinates system, the nodal points configuration in the left figure should correspond to the figure on the right hand. Where it implies the hexahedron of being projected to a cube with r_1-r_2-r_3 coordinates. This r_1-r_2-r_3 coordinates system is called natural coordinates system.

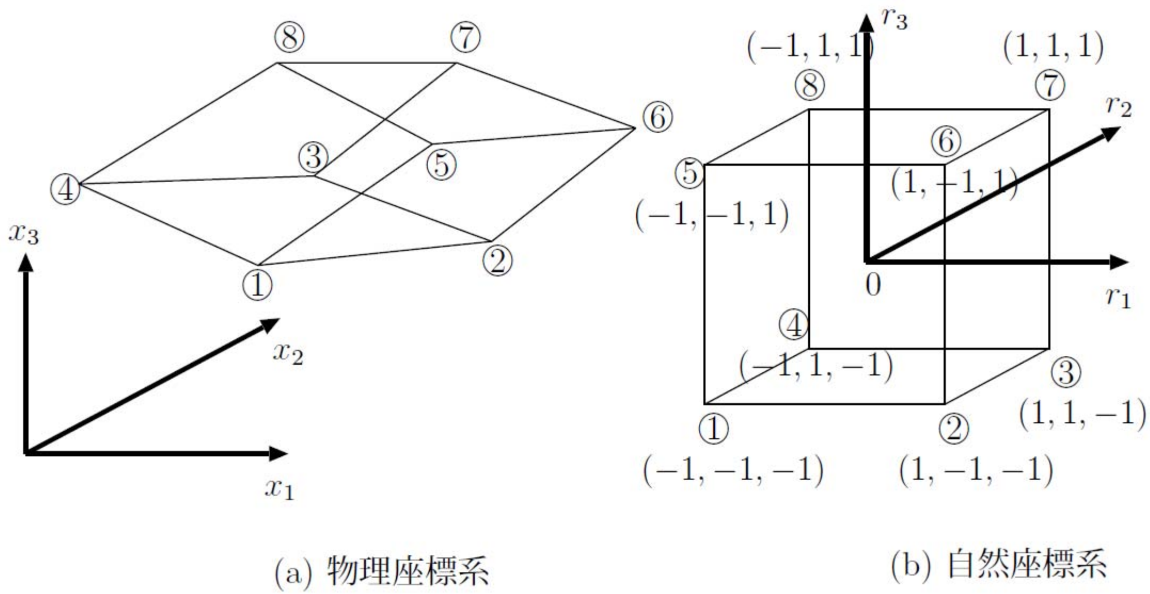


図 2: 物理座標系と自然座標系の対応

• Shape functions are expressed by,

$$N^{(1)} = \frac{1}{8}(1 - r_1)(1 - r_2)(1 - r_3) \tag{92}$$

$$N^{(2)} = \frac{1}{8}(1 + r_1)(1 - r_2)(1 - r_3) \tag{93}$$

$$N^{(3)} = \frac{1}{8}(1 + r_1)(1 + r_2)(1 - r_3) \tag{94}$$

$$N^{(4)} = \frac{1}{8}(1 - r_1)(1 + r_2)(1 - r_3) \tag{95}$$

$$N^{(5)} = \frac{1}{8}(1 - r_1)(1 - r_2)(1 + r_3) \tag{96}$$

$$N^{(6)} = \frac{1}{8}(1 + r_1)(1 - r_2)(1 + r_3) \tag{97}$$

$$N^{(7)} = \frac{1}{8}(1 + r_1)(1 + r_2)(1 + r_3) \tag{98}$$

$$N^{(8)} = \frac{1}{8}(1 - r_1)(1 + r_2)(1 + r_3) \tag{99}$$

Differentials in Discrete Expression 1

- Differentials of ui about xj, which are needed in calculating a strain, can be evaluated with chain rule in the following.

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial N^{(n)}}{\partial x_j} u_i^{(n)} = \left(\frac{\partial N^{(n)}}{\partial r_1}\frac{\partial r_1}{\partial x_j} + \frac{\partial N^{(n)}}{\partial r_2}\frac{\partial r_2}{\partial x_j} + \frac{\partial N^{(n)}}{\partial r_3}\frac{\partial r_3}{\partial x_j} \right) u_i^{(n)} \qquad (100)$$

- $\frac{\partial N^{(n)}}{\partial x_j}$ can be obtained also, with chain rule.

- Jacobian matrix $[J]$ may be found as,

$$\begin{bmatrix} \frac{\partial N^{(n)}}{\partial r_1} \\ \frac{\partial N^{(n)}}{\partial r_2} \\ \frac{\partial N^{(n)}}{\partial r_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_2}{\partial r_1} & \frac{\partial x_3}{\partial r_1} \\ \frac{\partial x_1}{\partial r_2} & \frac{\partial x_2}{\partial r_2} & \frac{\partial x_3}{\partial r_2} \\ \frac{\partial x_1}{\partial r_3} & \frac{\partial x_2}{\partial r_3} & \frac{\partial x_3}{\partial r_3} \end{bmatrix} \begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \\ \frac{\partial N^{(n)}}{\partial x_3} \end{bmatrix} = [J] \begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \\ \frac{\partial N^{(n)}}{\partial x_3} \end{bmatrix} \qquad (101)$$

Differentials in Discrete Expression 2

- Each component of this Jacobian matrix $\frac{\partial x_i}{\partial r_j}$ is given by,

$$\frac{\partial x_i}{\partial r_j} = \frac{\partial N^{(n)}}{\partial r_j} x_i^{(n)} \tag{102}$$

- $\frac{\partial N^{(n)}}{\partial x_i}$ is evaluated as,

$$\begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \\ \frac{\partial N^{(n)}}{\partial x_3} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial N^{(n)}}{\partial r_1} \\ \frac{\partial N^{(n)}}{\partial r_2} \\ \frac{\partial N^{(n)}}{\partial r_3} \end{bmatrix} \tag{103}$$

- In addition, the regional integration can be expressed by,

$$\int_{\Omega_e} d\Omega = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \det[J] dr_1\, dr_2\, dr_3 \tag{104}$$

- This integration is usually conducted by numerical integration method such as Gauss integration. Here, we simply use a summation triple Gauss integration in one-dimensional space.

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x,y,z)\, dx\, dy\, dz \approx \sum_i \sum_j \sum_k w_i w_j w_k f(x_i, y_j, z_k) \tag{105}$$

Lagrange Group

- In order to obtain more accuracy in single-dimension problems, the following interpolation functions are introduced by taking its nodal points within the segment of line.

$$N^{(1)} = -\frac{1}{2}r(1-r) \tag{106}$$

$$N^{(2)} = \frac{1}{2}r(1+r) \tag{107}$$

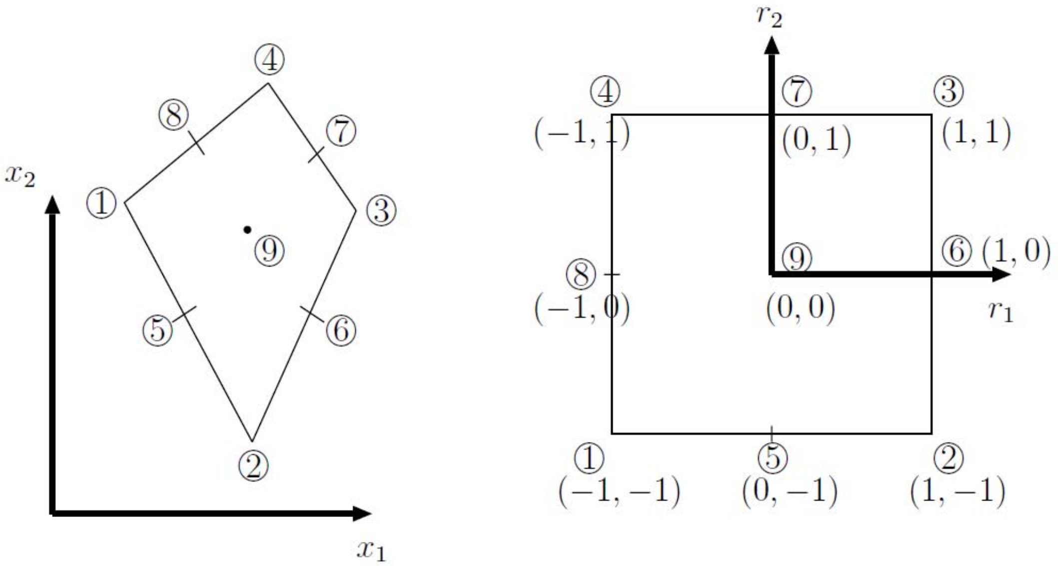
$$N^{(3)} = 1 - r^2 \tag{108}$$

- Instead of $r(-1 \leq r \leq 1)$, use two variables*r*1 and $r_1, r_2(-1 \leq r_1 \leq 1, -1 \leq r_2 \leq 1)$ multiplied by the interpolation functions to obtain 9 noded quadrilateral element.

節点	r_1	r_2
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0
9	0	0

表 3: 節点の対応

Correspondence of Physical Coordinates System and Natural Coordinates System



(a) 物理座標系　　(b) 自然座標系

図 3: 物理座標系と自然座標系の対応

Interpolation Function

- Specific forms in interpolation functions as well as its differentials are systematically expressed by considering single dimension.

$$N_{i-} = -\frac{1}{2}r_i(1-r_i) \tag{109}$$

$$N_{i0} = 1 - r_i^2 \tag{110}$$

$$N_{i+} = \frac{1}{2}r_i(1+r_i) \tag{111}$$

$$N^{(1)} = N_{1-}N_{2-} = \frac{1}{4}r_1 r_2(1-r_1)(1-r_2) \tag{112}$$

$$N^{(2)} = N_{1+}N_{2-} = -\frac{1}{4}r_1 r_2(1+r_1)(1-r_2) \tag{113}$$

$$N^{(3)} = N_{1+}N_{2+} = \frac{1}{4}r_1 r_2(1+r_1)(1+r_2) \tag{114}$$

$$N^{(4)} = N_{1-}N_{2+} = -\frac{1}{4}r_1 r_2(1-r_1)(1+r_2) \tag{115}$$

$$N^{(5)} = N_{10}N_{2-} = -\frac{1}{2}r_2(1-r_1^2)(1-r_2) \tag{116}$$

$$N^{(6)} = N_{1+}N_{20} = \frac{1}{2}r_1(1+r_1)(1-r_2^2) \tag{117}$$

$$N^{(7)} = N_{10}N_{2+} = \frac{1}{2}r_2(1-r_1^2)(1+r_2) \tag{118}$$

$$N^{(8)} = N_{1-}N_{20} = -\frac{1}{2}r_1(1-r_1)(1-r_2^2) \tag{119}$$

$$N^{(9)} = N_{10}N_{20} = (1-r_1^2)(1-r_2^2) \tag{120}$$

Origin of Lagrange Group

- In the discussion, it is verified that there is no difference in using hjgher-order functions as basis of single dimension interpolation function, and even an expansion to the three dimensions can be done easily. Such elements are often called Lagrange group.
- The name "Lagrange Group" originated in the common use of Lagrange polynomials in evaluating interpolation functions.

$$H_i(x) = \frac{(x - x_{(1)})(x - x_{(2)}) \ldots (x - x_{(i-1)})(x - x_{(i+1)}) \ldots (x - x_{(n)})}{(x_{(i)} - x_{(1)})(x_{(i)} - x_{(2)}) \ldots (x_{(i)} - x_{(i-1)})(x_{(i)} - x_{(i+1)}) \ldots (x_{(i)} - x_{(n)})} \tag{121}$$

- To actually obtain the single dimension interpolation functions by Lagrange polynomials,

  Where $n = 2$ $(x_{(1)} = -1, x_{(2)} = 1)$

$$H_1(x) = \frac{1}{2}(1 - x), \quad H_2(x) = \frac{1}{2}(1 + x) \tag{122}$$

Where $n = 3$ $(x_{(1)} = -1, x_{(2)} = 1, x_{(3)} = 0)$

$$H_1 = \frac{1}{2}x(x - 1), \quad H_2 = \frac{1}{2}x(x + 1), \quad H_3 = 1 - x^2 \tag{123}$$

Where $n = 4$ $(x_{(1)} = -1, x_{(2)} = 1, x_{(3)} = -\frac{1}{3}, x_{(4)} = \frac{1}{3})$

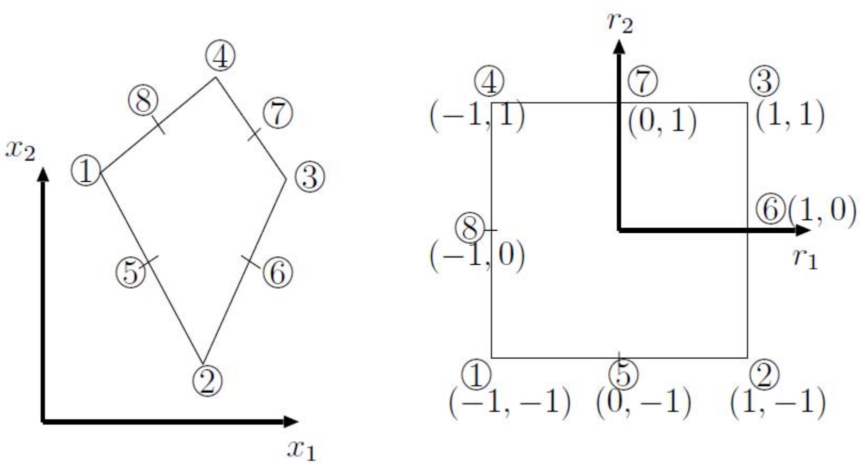
$$\begin{aligned} H_1 &= -\frac{1}{16}(x - 1)(9x^2 - 1), \quad H_2 = \frac{1}{16}(x + 1)(9x^2 - 1) \\ H_3 &= \frac{9}{16}(x^2 - 1)(3x - 1), \quad H_4 = -\frac{9}{16}(x^2 - 1)(3x + 1) \end{aligned} \tag{124}$$

- However, in general, the interpolation functions under two- and three- dimensional spaces with $n = 3<$ are rarely used for its complication in calculation.

Serendipity Group

- Consider now for an element that includes nodal points only on the sides of a rectangular.
- This element is called serendipity group. 8 nodal points are found in rectangular with the specific forms in the interpolation functions to be the following,

節点	r_1	r_2	節点	r_1	r_2
1	-1	-1	5	0	-1
2	1	-1	6	1	0
3	1	1	7	0	1
4	-1	1	8	-1	0



(a) 物理座標系　　(b) 自然座標系

Interpolation Functions in Serendipity Group 1

$$N^{(1)} = \frac{1}{4}(1 - r_1)(1 - r_2)(-1 - r_1 - r_2) \tag{125}$$

$$N^{(2)} = \frac{1}{4}(1 + r_1)(1 - r_2)(-1 + r_1 - r_2) \tag{126}$$

$$N^{(3)} = \frac{1}{4}(1 + r_1)(1 + r_2)(-1 + r_1 + r_2) \tag{127}$$

$$N^{(4)} = \frac{1}{4}(1 - r_1)(1 + r_2)(-1 - r_1 + r_2) \tag{128}$$

$$N^{(5)} = \frac{1}{2}(1 - r_2)(1 - r_1^2) \tag{129}$$

$$N^{(6)} = \frac{1}{2}(1 + r_1)(1 - r_2^2) \tag{130}$$

$$N^{(7)} = \frac{1}{2}(1 + r_2)(1 - r_1^2) \tag{131}$$

$$N^{(8)} = \frac{1}{2}(1 - r_1)(1 - r_2^2) \tag{132}$$

- Because the way this interpolation function was found so accidentally, it is called this way since then.

Interpolation Functions in Serendipity Group 2

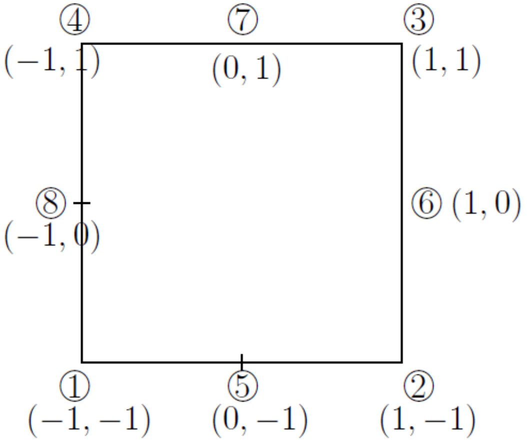
- Consider now, with the interpolation functions for 4 nodal points, for obtaining the interpolation functions for 8 nodal points. For the 4 nodal points, the functions are expressed by,

$$N^{(1)} = \frac{1}{4}(1 - r_1)(1 - r_2) \tag{133}$$

$$N^{(2)} = \frac{1}{4}(1 + r_1)(1 - r_2) \tag{134}$$

$$N^{(3)} = \frac{1}{4}(1 + r_1)(1 + r_2) \tag{135}$$

$$N^{(4)} = \frac{1}{4}(1 - r_1)(1 + r_2) \tag{136}$$



Value being found as 1 at corresponding nodal points, while 0 is found in other nodal points.

Interpolation Functions in Serendipity Group 3

- As for the 5 th nodal points, consider now the points $(0,-1)$ in natural coordinates system.

$$N^{(5)} = \frac{1}{2}(1-r_2)(1-r_1^2) \tag{137}$$

At point 5, we find the value to be1, while the points 1 - 4, the value is found as 0. In $N^{(1)} \sim N^{(4)}$ the point 5 will not be 0, thus we can write the interpolation functions in new way.

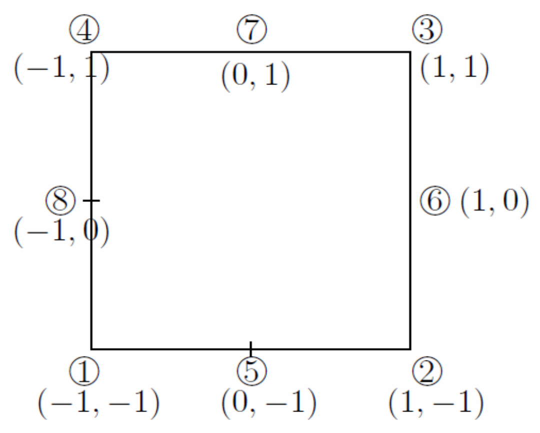
$$\tilde{N}^{(1)} = N^{(1)} - \frac{1}{2}N^{(5)} \tag{138}$$

$$\tilde{N}^{(2)} = N^{(2)} - \frac{1}{2}N^{(5)} \tag{139}$$

$$\tilde{N}^{(3)} = N^{(3)} \tag{140}$$

$$\tilde{N}^{(4)} = N^{(4)} \tag{141}$$

- Here , $\tilde{N}^{(1)} \sim \tilde{N}^{(4)}$ takes the value 1 at corresponding nodal points while in the rest of the nodal points including 5, the value is found to be zero.



Interpolation Functions in Serendipity Group 4

- In the same way, consider the 6th 7th and 8th nodal points $(1, 0)$, $(0, 1)$and $(-1, 0)$. The corresponding interpolation functions are in the following.

$$N^{(6)} = \frac{1}{2}(1 + r_1)(1 - r_2^2) \tag{142}$$

$$N^{(7)} = \frac{1}{2}(1 + r_2)(1 - r_1^2) \tag{143}$$

$$N^{(8)} = \frac{1}{2}(1 - r_1)(1 - r_2^2) \tag{144}$$

- Then,

$$\tilde{N}^{(1)} = N^{(1)} - \frac{1}{2}N^{(5)} - \frac{1}{2}N^{(8)} \tag{145}$$

$$\tilde{N}^{(2)} = N^{(2)} - \frac{1}{2}N^{(5)} - \frac{1}{2}N^{(6)} \tag{146}$$

$$\tilde{N}^{(3)} = N^{(3)} - \frac{1}{2}N^{(6)} - \frac{1}{2}N^{(7)} \tag{147}$$

$$\tilde{N}^{(4)} = N^{(4)} - \frac{1}{2}N^{(7)} - \frac{1}{2}N^{(8)} \tag{148}$$

At corresponding nodal points, we obtain 1, but for the rest of other points, we obtain the value 0. To actually calculate this fact, which is expressed by,

$$N^{(1)} = \frac{1}{4}(1 - r_1)(1 - r_2)(-1 - r_1 - r_2) \tag{149}$$

$$N^{(2)} = \frac{1}{4}(1 + r_1)(1 - r_2)(-1 + r_1 - r_2) \tag{150}$$

$$N^{(3)} = \frac{1}{4}(1 + r_1)(1 + r_2)(-1 + r_1 + r_2) \tag{151}$$

$$N^{(4)} = \frac{1}{4}(1 - r_1)(1 + r_2)(-1 - r_1 + r_2) \tag{152}$$

Which coincides with the shape function stated in the 8 noded element.

Interpolation Functions in Serendipity Group 5

- Furthermore, we can obtain 9 noded element through 8 noded element. When we take the 9th nodal points at (0,0),

$$N^{(9)} = (1 - r_1^2)(1 - r_2^2) \tag{153}$$

Then, obtain 1 at corresponding nodal points but in the rest of other points, we obtain 0. Define $N^{(1)} \sim N^{(8)}$ to write as the following so, we can obtain the interpolation function for 9 nodal points in Lagrange group.

$$N^{(1)} = \frac{1}{4}(1 - r_1)(1 - r_2)(-1 - r_1 - r_2) + \frac{1}{4}N^{(9)} \tag{154}$$

$$N^{(2)} = \frac{1}{4}(1 + r_1)(1 - r_2)(-1 + r_1 - r_2) + \frac{1}{4}N^{(9)} \tag{155}$$

$$N^{(3)} = \frac{1}{4}(1 + r_1)(1 + r_2)(-1 + r_1 + r_2) + \frac{1}{4}N^{(9)} \tag{156}$$

$$N^{(4)} = \frac{1}{4}(1 - r_1)(1 + r_2)(-1 - r_1 + r_2) + \frac{1}{4}N^{(9)} \tag{157}$$

$$N^{(5)} = \frac{1}{2}(1 - r_2)(1 - r_1^2) - \frac{1}{2}N^{(9)} \tag{158}$$

$$N^{(6)} = \frac{1}{2}(1 + r_1)(1 - r_2^2) - \frac{1}{2}N^{(9)} \tag{159}$$

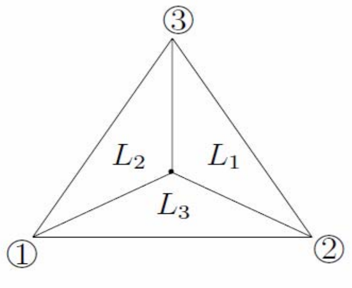
$$N^{(7)} = \frac{1}{2}(1 + r_2)(1 - r_1^2) - \frac{1}{2}N^{(9)} \tag{160}$$

$$N^{(8)} = \frac{1}{2}(1 - r_1)(1 - r_2^2) - \frac{1}{2}N^{(9)} \tag{161}$$

Interpolation Functions For Triangle Element 1

- Interpolation functions in triangle element are expressed in the area coordinates defined by the following.
- Area coordinates represent the coordinates consisted of the area of element A ,and the given points within the element. In addition, the area of triangles are given A_1, A_2, A_3 (triangles made by the corresponding opposite sides of nodal points and its points)

$$\begin{aligned} L_1 &= A_1/A \\ L_2 &= A_2/A \\ L_3 &= A_3/A \end{aligned} \tag{162}$$



$$L_1 + L_2 + L_3 = 1 \tag{163}$$

Interpolation Functions For Triangle Element 2

- Interpolation functions in single dimension with 3 nodal points

$$N^{(1)} = L_1 \tag{164}$$

$$N^{(2)} = L_2 \tag{165}$$

$$N^{(3)} = L_3 \tag{166}$$

- Interpolation functions in the two-dimensional 6 nodes,

$$N^{(1)} = L_1(2L_1 - 1) \tag{167}$$

$$N^{(2)} = L_2(2L_2 - 1) \tag{168}$$

$$N^{(3)} = L_3(2L_3 - 1) \tag{169}$$

$$N^{(4)} = 4L_2L_3 \tag{170}$$

$$N^{(5)} = 4L_3L_1 \tag{171}$$

$$N^{(6)} = 4L_1L_2 \tag{172}$$

- We can obtain the 6 nodes interpolation functions through 3 nodes functions.

Interpolation Functions and Numerical Analysis for Triangle Element 1

- In actual calculations for element stiffness matrix, the numerical integration is necessary.
- Numerical integration is conducted by reflecting the area coordinates L_1, L_2, L_3 and the natural coordinates system r_1, r_2 in the way shows in the following.

$$r_1 = L_1 \tag{173}$$

$$r_2 = L_2 \tag{174}$$

$$1 - r_1 - r_2 = L_3 \tag{175}$$

- Domain for the triangle internal corresponds to the domain for the natural coordinates system appears in the figure below.

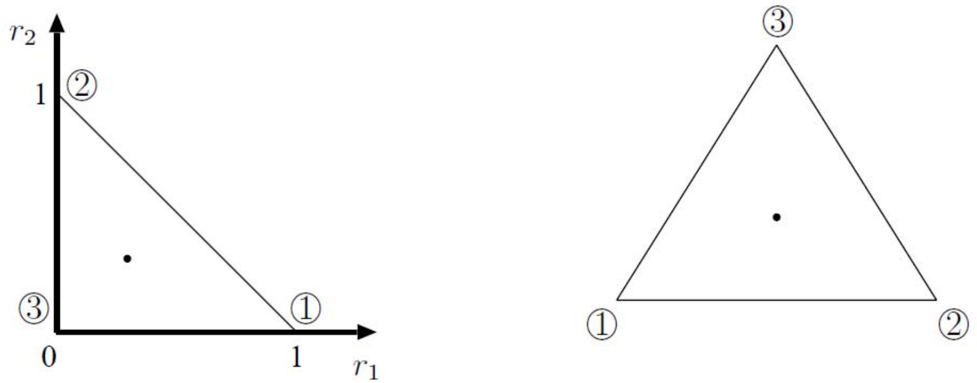


図 4: 三角形要素の自然座標と面積座標

Interpolation Functions and Numerical Analysis for Triangle Element 2

- Under physical space, integral $\int_V dV$ transforms into the natural coordinates system by Jacobian matrix, in the same way we evaluated for the rectangular element.

$$\begin{bmatrix} \dfrac{\partial N^{(i)}}{\partial r_1} \\ \dfrac{\partial N^{(i)}}{\partial r_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x_1}{\partial r_1} & \dfrac{\partial x_2}{\partial r_1} \\ \dfrac{\partial x_1}{\partial r_2} & \dfrac{\partial x_2}{\partial r_2} \end{bmatrix}}_{\substack{\Downarrow \\ [J]}} \begin{bmatrix} \dfrac{\partial N^{(i)}}{\partial x_1} \\ \dfrac{\partial N^{(i)}}{\partial x_2} \end{bmatrix} \tag{176}$$

- Thus,

$$\begin{bmatrix} \dfrac{\partial N^{(i)}}{\partial x_1} \\ \dfrac{\partial N^{(i)}}{\partial x_2} \end{bmatrix} = [J^{-1}] \begin{bmatrix} \dfrac{\partial N^{(i)}}{\partial r_1} \\ \dfrac{\partial N^{(i)}}{\partial r_2} \end{bmatrix} \tag{177}$$

- Jacobian matrix component $\dfrac{\partial x_i}{\partial r_j}$ becomes what we obtained for the rectangular element in the following.

$$\frac{\partial x_i}{\partial r_j} = \frac{\partial N^{(n)}}{\partial r_j} x_i^{(n)} \tag{178}$$

Interpolation Functions and Numerical Analysis for Triangle Element 3

- Here, a differential $\frac{\partial N^{(i)}}{\partial r_j}$ for shape functions by natural coordinates appears,

$$r_1 = L_1 \tag{179}$$

$$r_2 = L_2 \tag{180}$$

$$1 - r_1 - r_2 = L_3 \tag{181}$$

Based on the functions above, reflect with the area coordinates to obtain,

$$\frac{\partial N^{(i)}}{\partial r_1} = \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_1} \tag{182}$$

$$= \frac{\partial N^{(i)}}{\partial L_1} - \frac{\partial N^{(i)}}{\partial L_3} \tag{183}$$

$$\frac{\partial N^{(i)}}{\partial r_2} = \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_2} \tag{184}$$

$$= \frac{\partial N^{(i)}}{\partial L_2} - \frac{\partial N^{(i)}}{\partial L_3} \tag{185}$$

Interpolation Functions and Numerical Analysis for Triangle Element 4

- In respect, conduct $\int_V dV \Rightarrow \int_0^1 \int_0^{1-r_1} \det J \, dr_2 \, dr_1$

Apparently in the form $\iint F \, dr_1 \, dr_2 = \frac{1}{2} \sum w_i F(x_i, y_i)$

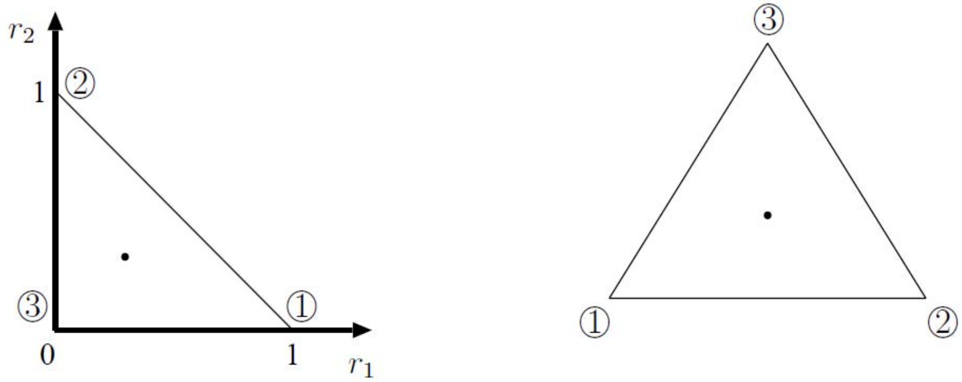


図 5: 三角形要素の自然座標と面積座標

Interpolation Functions and Numerical Analysis for Triangle Element 5

Integration order	Degree of precision	r-coordinates	s-coordinates	Weights
3-point	2	$r_1 = 0.16666\ 66666\ 667$ $r_2 = 0.66666\ 66666\ 667$ $r_3 = r_1$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$	$w_1 = 0.33333\ 33333\ 333$ $w_2 = w_1$ $w_3 = w_1$
7-point	5	$r_1 = 0.10128\ 65073\ 235$ $r_2 = 0.79742\ 69853\ 531$ $r_3 = r_1$ $r_4 = 0.47014\ 20641\ 051$ $r_5 = r_4$ $r_6 = 0.05971\ 58717\ 898$ $r_7 = 0.33333\ 33333\ 333$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$ $s_4 = r_6$ $s_5 = r_4$ $s_6 = r_4$ $s_7 = r_7$	$w_1 = 0.12593\ 91805\ 448$ $w_2 = w_1$ $w_3 = w_1$ $w_4 = 0.13239\ 41527\ 885$ $w_5 = w_4$ $w_6 = w_4$ $w_7 = 0.225$
13-point	7	$r_1 = 0.06513\ 01029\ 002$ $r_2 = 0.86973\ 97941\ 956$ $r_3 = r_1$ $r_4 = 0.31286\ 54960\ 049$ $r_5 = 0.63844\ 41885\ 698$ $r_6 = 0.04869\ 03154\ 253$ $r_7 = r_5$ $r_8 = r_4$ $r_9 = r_6$ $r_{10} = 0.26034\ 59660\ 790$ $r_{11} = 0.47930\ 80678\ 419$ $r_{12} = r_{10}$ $r_{13} = 0.33333\ 33333\ 333$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$ $s_4 = r_6$ $s_5 = r_4$ $s_6 = r_5$ $s_7 = r_6$ $s_8 = r_5$ $s_9 = r_4$ $s_{10} = r_{10}$ $s_{11} = r_{10}$ $s_{12} = r_{11}$ $s_{13} = r_{13}$	$w_1 = 0.05334\ 72356\ 008$ $w_2 = w_1$ $w_3 = w_1$ $w_4 = 0.07711\ 37608\ 903$ $w_5 = w_4$ $w_6 = w_4$ $w_7 = w_4$ $w_8 = w_4$ $w_9 = w_4$ $w_{10} = 0.17561\ 52574\ 332$ $w_{11} = w_{10}$ $w_{12} = w_{10}$ $w_{13} = -0.14957\ 00444\ 67$

Interpolation Functions for Tetrahedral Element 1

- Interpolation functions for tetrahedral can be expressed in the volume coordinates defined by the following.
- Volume coordinates represent the volume of triangular pyramid A_1, A_2, A_3, A_4 made by the element with volume A, with its internal points given to form a triangular pyramid with corresponding nodal points opposite sides and its points, and which is defined as following,

$$
\begin{aligned}
L_1 &= A_1/A \\
L_2 &= A_2/A \\
L_3 &= A_3/A \\
L_4 &= A_4/A
\end{aligned}
\tag{186}
$$

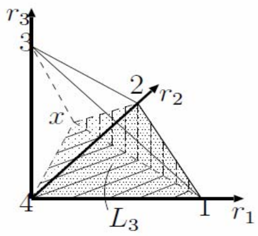


図 6: 体積座標系

$$
L_1 + L_2 + L_3 + L_4 = 1 \tag{187}
$$

Interpolation Functions for Tetrahedral Element 2

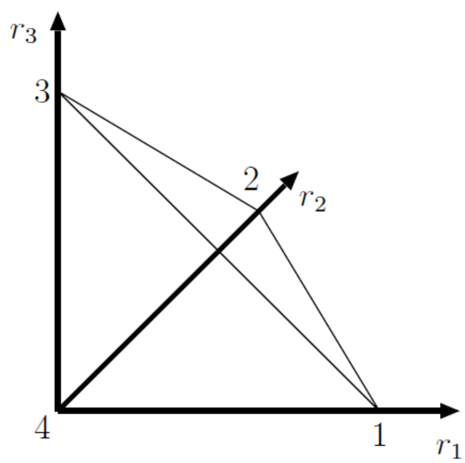
- Linear 4 nodes interpolation functions are,

$$N^{(1)} = L_1 \tag{188}$$

$$N^{(2)} = L_2 \tag{189}$$

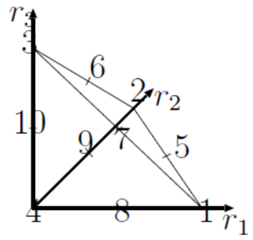
$$N^{(3)} = L_3 \tag{190}$$

$$N^{(4)} = L_4 \tag{191}$$



Interpolation Functions for Tetrahedral Element 3

- The second 10 nodes interpolation function



$$N^{(5)} = 4L_1L_2 \tag{192}$$

$$N^{(6)} = 4L_2L_3 \tag{193}$$

$$N^{(7)} = 4L_1L_3 \tag{194}$$

$$N^{(8)} = 4L_1L_4 \tag{195}$$

$$N^{(9)} = 4L_2L_4 \tag{196}$$

$$N^{(10)} = 4L_3L_4 \tag{197}$$

$$N^{(1)} = N^{(1)} - \frac{1}{2}(N^{(5)} + N^{(7)} + N^{(8)}) \ = L_1(2L_1 - 1) \tag{198}$$

$$N^{(2)} = N^{(2)} - \frac{1}{2}(N^{(5)} + N^{(6)} + N^{(9)}) \ = L_2(2L_2 - 1) \tag{199}$$

$$N^{(3)} = N^{(3)} - \frac{1}{2}(N^{(6)} + N^{(7)} + N^{(10)}) = L_3(2L_3 - 1) \tag{200}$$

$$N^{(4)} = N^{(4)} - \frac{1}{2}(N^{(8)} + N^{(9)} + N^{(10)}) = L_4(2L_4 - 1) \tag{201}$$

Interpolation Functions and Numerical Integrations for Tetrahedral Element 1

- In actual calculation of element stiffness matrix, numerical integration become necessary.
- Numerical integration is conducted by reflecting this volume coordinates L_1, L_2, L_3, l_4 to the natural coordinates system r_1, r_2, r_3 ,

$$L_1 = r_1 \tag{202}$$

$$L_2 = r_2 \tag{203}$$

$$L_3 = r_3 \tag{204}$$

$$L_4 = 1 - r_1 - r_2 - r_3 \tag{205}$$

- Integral $\int_V dV$ in physical space can be transformed by Jacobian matrix in the same way with hexahedral element.

$$\begin{bmatrix} \frac{\partial N^{(i)}}{\partial r_1} \\ \frac{\partial N^{(i)}}{\partial r_2} \\ \frac{\partial N^{(i)}}{\partial r_3} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial r_1} & \frac{\partial y}{\partial r_1} & \frac{\partial z}{\partial r_1} \\ \frac{\partial x}{\partial r_2} & \frac{\partial y}{\partial r_2} & \frac{\partial z}{\partial r_2} \\ \frac{\partial x}{\partial r_3} & \frac{\partial y}{\partial r_3} & \frac{\partial z}{\partial r_3} \end{bmatrix}}_{\Downarrow \atop [J]} \begin{bmatrix} \frac{\partial N^{(i)}}{\partial x_1} \\ \frac{\partial N^{(i)}}{\partial x_2} \\ \frac{\partial N^{(i)}}{\partial x_3} \end{bmatrix} \quad \begin{bmatrix} \frac{\partial N^{(i)}}{\partial x_1} \\ \frac{\partial N^{(i)}}{\partial x_2} \\ \frac{\partial N^{(i)}}{\partial x_3} \end{bmatrix} = [J^{-1}] \begin{bmatrix} \frac{\partial N^{(i)}}{\partial r_1} \\ \frac{\partial N^{(i)}}{\partial r_2} \\ \frac{\partial N^{(i)}}{\partial r_3} \end{bmatrix} \tag{206}$$

$$\frac{\partial x_i}{\partial r_j} = \frac{\partial N^{(n)}}{\partial r_j} x_i^{(n)} \tag{207}$$

- Jacobian matrix components $\frac{\partial x_i}{\partial r_j}$ becomes as following.

Interpolation Functions and Numerical Integrations for Tetrahedral Element 2

• Differential $\frac{\partial N^{(i)}}{\partial r_j}$ for the shape functions by natural coordinates appears,

$$L_1 = r_1 \tag{208}$$

$$L_2 = r_2 \tag{209}$$

$$L_3 = r_3 \tag{210}$$

$$L_4 = 1 - r_1 - r_2 - r_3 \tag{211}$$

Based on above, reflect them with the area coordinates,

$$\begin{aligned}
\frac{\partial N^{(i)}}{\partial r_1} &= \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_4}\frac{\partial L_4}{\partial r_1} \\
&= \frac{\partial N^{(i)}}{\partial L_1} - \frac{\partial N^{(i)}}{\partial L_4} \\
\frac{\partial N^{(i)}}{\partial r_2} &= \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_4}\frac{\partial L_4}{\partial r_2} \\
&= \frac{\partial N^{(i)}}{\partial L_2} - \frac{\partial N^{(i)}}{\partial L_4} \\
\frac{\partial N^{(i)}}{\partial r_3} &= \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_3} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_3} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_3} + \frac{\partial N^{(i)}}{\partial L_4}\frac{\partial L_4}{\partial r_3} \\
&= \frac{\partial N^{(i)}}{\partial L_3} - \frac{\partial N^{(i)}}{\partial L_4}
\end{aligned} \tag{212}$$

Interpolation Functions and Numerical Integrations for Tetrahedral Element 3

$$\int_V F(x,y,z)dV = \frac{1}{6}\sum w_i F(x_i, y_i, z_i) \tag{213}$$

Conduct numerical integration in the form above.

- Integration points and the weight

N	M	Weight W	α	β	ν	δ
2	4	$0.25000\ 00000\ 000000^{+000}$	$0.58541^{01966}249685^{+000}$	$0.13819^{66011}25015^{+000}$	$0.13819^{66011}250105^{+000}$	$0.13819^{66011}250105^{+000}$
3	1	$-0.80000^{00000}000000^{+000}$	$0.25000^{00000}000000^{+000}$	$0.25000^{00000}000000^{+000}$	$0.25000^{00000}000000^{+000}$	$0.25000^{00000}000000^{+000}$
	4	$0.45000^{00000}000000^{+000}$	$0.50000^{00000}000000^{+000}$	$0.16666^{66666}666667^{+000}$	$0.16666^{66666}666667^{+000}$	$0.16666^{66666}666667^{+000}$
4	4	$0.50373\ 79410\ 012282^{-001}$	$0.77164\ 29020\ 672371^{+000}$	$0.76119\ 03264\ 425430^{-001}$	$0.76119\ 03264\ 425430^{-001}$	$0.76119\ 03264\ 425430^{-001}$
	12	$0.66542\ 06863\ 329239^{-001}$	$0.11970\ 05277\ 978019^{+000}$	$0.71831\ 64526\ 766925^{-001}$	$0.40423\ 39134\ 672644^{+000}$	$0.40423\ 39134\ 672644^{+000}$
5	1	$0.18841\ 85567\ 365411^{+000}$	$0.25000\ 00000\ 000000^{+000}$	$0.25000\ 00000\ 000000^{+000}$	$0.25000\ 00000\ 000000^{+000}$	$0.25000\ 00000\ 000000^{+000}$
	4	$0.67038\ 58372\ 604275^{-001}$	$0.73163\ 69079\ 576180^{+000}$	$0.89454\ 36401\ 412733^{-001}$	$0.89454\ 36401\ 412733^{-001}$	$0.89454\ 36401\ 412733^{-001}$
	12	$0.45285\ 59236\ 327399^{-001}$	$0.13258\ 10999\ 384657^{+000}$	$0.24540\ 03792\ 903000^{-001}$	$0.42143\ 94310\ 662522^{+000}$	$0.42143\ 94310\ 662522^{+000}$
6	1	$0.90401\ 29046\ 014750^{-001}$	$0.25000\ 00000\ 000000^{+000}$	$0.25000\ 00000\ 000000^{+000}$	$0.25000\ 00000\ 000000^{+000}$	$0.25000\ 00000\ 000000^{+000}$
	4	$0.19119\ 83427\ 899124^{-001}$	$0.82771\ 92480\ 479295^{+000}$	$0.57426\ 91731\ 735683^{-001}$	$0.57426\ 91731\ 735683^{-001}$	$0.57426\ 91731\ 735683^{-001}$
	12	$0.43614\ 93840\ 666568^{-001}$	$0.51351\ 88412\ 556341^{-001}$	$0.48605\ 10285\ 706072^{+000}$	$0.23129\ 85436\ 519147^{+000}$	$0.23129\ 85436\ 519147^{+000}$
	12	$0.25811\ 67596\ 199161^{-001}$	$0.29675\ 38129\ 690260^{+000}$	$0.60810\ 79894\ 015281^{+000}$	$0.47569\ 09881\ 472290^{-001}$	$0.47569\ 09881\ 472290^{-001}$

Drawing the Stiffness Matrix– 4Noded Quadrilateral 1

- In finding element stiffness matrix for two-dimensional 4 noded quadrilateral element by 2 ×2 Gauss integration,
- Transform the domain of integration of the element stiffness matrix.（transformation of the domain of integration by isoparametric element)

$$[K^{(e)}] = \int_{\Omega_e} [B]^T [D] [B] \, \mathrm{d}\Omega = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] \det[J] dr_2 dr_1 \tag{214}$$

$$[J] = \begin{bmatrix} \dfrac{\partial x_1}{\partial r_1} & \dfrac{\partial x_2}{\partial r_1} \\ \dfrac{\partial x_1}{\partial r_2} & \dfrac{\partial x_2}{\partial r_2} \end{bmatrix} \tag{215}$$

- Introduce the numerical integration to yield,

$$\int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] \det[J] dr_2 dr_1$$
$$= \sum_{a=1}^{2} \sum_{b=1}^{2} w_a \, w_b \, [B(r_a, r_b)]^T [D] [B(r_a, r_b)] \det[J(r_a, r_b)] \tag{216}$$

Drawing the Stiffness Matrix– 4Noded Quadrilateral 2

- In specific, components of Jacobian matrix can be evaluated by setting each sampling point r_a, r_b in the following,

$$\frac{\partial x_i}{\partial r_j}(r_a, r_b) = \frac{\partial N^{(n)}}{\partial r_j}(r_a, r_b)\, x_i^{(n)}$$
$$= \frac{\partial N^{(1)}}{\partial r_j}(r_a, r_b)\, x_i^{(1)} + \frac{\partial N^{(2)}}{\partial r_j}(r_a, r_b)\, x_i^{(2)} + \frac{\partial N^{(3)}}{\partial r_j}(r_a, r_b)\, x_i^{(3)} + \frac{\partial N^{(4)}}{\partial r_j}(r_a, r_b)\, x_i^{(4)} \quad (217)$$

$$\frac{\partial N^{(1)}}{\partial r_1} = -\frac{1}{4}(1 - r_2) \quad (218) \qquad \frac{\partial N^{(3)}}{\partial r_1} = \frac{1}{4}(1 + r_2) \quad (222)$$
$$\frac{\partial N^{(1)}}{\partial r_2} = -\frac{1}{4}(1 - r_1) \quad (219) \qquad \frac{\partial N^{(3)}}{\partial r_2} = \frac{1}{4}(1 + r_1) \quad (223)$$
$$\frac{\partial N^{(2)}}{\partial r_1} = \frac{1}{4}(1 - r_2) \quad (220) \qquad \frac{\partial N^{(4)}}{\partial r_1} = -\frac{1}{4}(1 + r_2) \quad (224)$$
$$\frac{\partial N^{(2)}}{\partial r_2} = -\frac{1}{4}(1 + r_1) \quad (221) \qquad \frac{\partial N^{(4)}}{\partial r_2} = \frac{1}{4}(1 - r_1) \quad (225)$$

- Clarify all components in the matrix and again, draw out the Jacobian matrix, then follow through the steps to complete the calculation.

Drawing the Stiffness Matrix– 4Noded Quadrilateral 3

- $[B]$ Matrix components are obtained by following

$$\begin{bmatrix} \dfrac{\partial N^{(n)}}{\partial x_1} \\ \dfrac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \dfrac{\partial N^{(n)}}{\partial r_1} \\ \dfrac{\partial N^{(n)}}{\partial r_2} \end{bmatrix} \tag{226}$$

- $[B]$ Substitute each value into the corresponding part in matrix.

$$[B^{(k)}] = \begin{bmatrix} \dfrac{\partial N^{(k)}}{\partial x_1} & \\ & \dfrac{\partial N^{(k)}}{\partial x_2} \\ \dfrac{\partial N^{(k)}}{\partial x_2} & \dfrac{\partial N^{(k)}}{\partial x_1} \end{bmatrix} \tag{227}$$

$$[B] = \left[[B^{(1)}], [B^{(2)}], \ldots, [B^{(n)}] \right] \tag{228}$$

- From above, $[B(r_a, r_b)] \det[J(r_a, r_b)]$ is gained,

$$[B(r_a, r_b)]^T [D] [B(r_a, r_b)] \det[J(r_a, r_b)] \tag{229}$$

Calculate the above then multiply the weight $w_a w_b$ then plug them into the configuration of the total stiffness matrix,

$$\begin{aligned} [K^{(e)}] &= \int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] \det[J] dr_1 dr_2 \\ &= \sum_{a=1}^{2} \sum_{b=1}^{2} w_a w_b [B(r_a, r_b)]^T [D] [B(r_a, r_b)] \det[J(r_a, r_b)] \end{aligned} \tag{230}$$

Drawing the Stiffness Matrix– Triangle 1

- For the triangle element, basically, we can take the same steps.
- Domain of integration in the element stiffness matrix is transformed as the figure indicates]
（transformation of integration domain by isoparametric element)

$$
\begin{aligned}
[K^{(e)}] &= \int_{\Omega_e} [B]^T \, [D] \, [B] \, \mathrm{d}\Omega \\
&= \int_0^1 \int_0^{1-r_1} [B]^T \, [D] \, [B] \det[J] dr_2 dr_1
\end{aligned} \tag{231}
$$

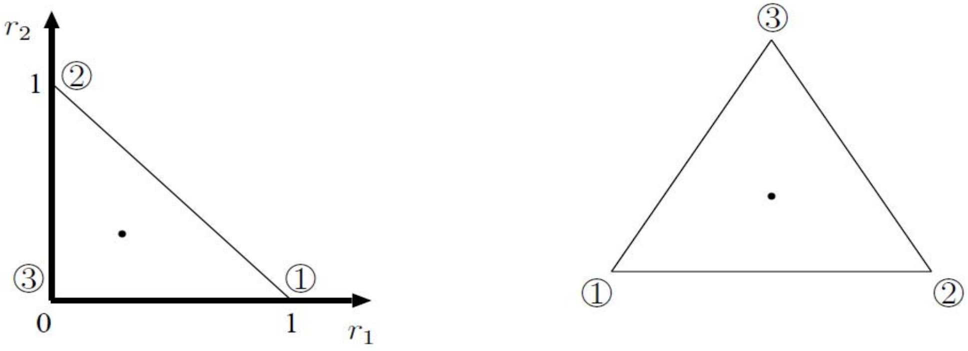


図 7: 三角形要素の自然座標と面積座標

$$
[J] = \begin{bmatrix} \dfrac{\partial x_1}{\partial r_1} & \dfrac{\partial x_2}{\partial r_1} \\ \dfrac{\partial x_1}{\partial r_2} & \dfrac{\partial x_2}{\partial r_2} \end{bmatrix}
$$

Drawing the Stiffness Matrix– Triangle 2

- Introduce the numerical integration

$$\int_0^1 \int_0^{1-r_1} [B]^T \, [D] \, [B] \det[J] dr_2 dr_1 = \frac{1}{2} \sum_{a,b=1}^{n} w_{ab} \, [B(r_a, r_b)]^T \, [D] \, [B(r_a, r_b)] \det[J(r_a, r_b)] \quad (233)$$

- Evaluate the values for each sampling point to draw the matrix, then plug them into the total stiffness matrix.

$$\int_0^1 \int_0^{1-r_1} [B]^T \, [D] \, [B] \det[J] dr_2 dr_1 = \frac{1}{2} \sum_{a,b=1}^{n} w_{ab} \, [B(r_a, r_b)]^T \, [D] \, [B(r_a, r_b)] \det[J(r_a, r_b)] \quad (234)$$

- Evaluate the values for each sampling point to draw the matrix, then plug them into the total stiffness matrix.

Drawing the Stiffness Matrix– Triangle 3

- Interpolation functions for 3 noded triangle element can be written by,

$$N^{(1)} = L_1 \tag{235}$$
$$N^{(2)} = L_2 \tag{236}$$
$$N^{(3)} = L_3 \tag{237}$$

- Base on this relations, reflect it to the area coordinates system in the following,

$$\frac{\partial N^{(i)}}{\partial r_1} = \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_1} = \frac{\partial N^{(i)}}{\partial L_1} - \frac{\partial N^{(i)}}{\partial L_3} \tag{238}$$
$$\frac{\partial N^{(i)}}{\partial r_2} = \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_2} = \frac{\partial N^{(i)}}{\partial L_2} - \frac{\partial N^{(i)}}{\partial L_3} \tag{239}$$

- For the calculation detail,

$$\begin{aligned} \frac{\partial N^{(1)}}{\partial r_1} &= 1 \quad \frac{\partial N^{(2)}}{\partial r_1} = 0 \quad \frac{\partial N^{(3)}}{\partial r_1} = -1 \\ \frac{\partial N^{(1)}}{\partial r_2} &= 0 \quad \frac{\partial N^{(2)}}{\partial r_2} = 1 \quad \frac{\partial N^{(3)}}{\partial r_2} = -1 \end{aligned} \tag{240}$$

Drawing the Stiffness Matrix– Triangle 4

- Jacobian matrix components, $\frac{\partial x_i}{\partial r_j}$ are,

$$
\begin{aligned}
\frac{\partial x_1}{\partial r_1} &= \frac{\partial N^{(1)}}{\partial r_1}x_1^{(1)} + \frac{\partial N^{(2)}}{\partial r_1}x_1^{(2)} + \frac{\partial N^{(3)}}{\partial r_1}x_1^{(3)} = x_1^{(1)} - x_1^{(3)} \\
\frac{\partial x_2}{\partial r_1} &= \frac{\partial N^{(1)}}{\partial r_1}x_2^{(1)} + \frac{\partial N^{(2)}}{\partial r_1}x_2^{(2)} + \frac{\partial N^{(3)}}{\partial r_1}x_2^{(3)} = x_2^{(1)} - x_2^{(3)} \\
\frac{\partial x_1}{\partial r_2} &= \frac{\partial N^{(1)}}{\partial r_2}x_1^{(1)} + \frac{\partial N^{(2)}}{\partial r_2}x_1^{(2)} + \frac{\partial N^{(3)}}{\partial r_2}x_1^{(3)} = x_1^{(2)} - x_1^{(3)} \\
\frac{\partial x_2}{\partial r_2} &= \frac{\partial N^{(1)}}{\partial r_2}x_2^{(1)} + \frac{\partial N^{(2)}}{\partial r_2}x_2^{(2)} + \frac{\partial N^{(3)}}{\partial r_2}x_2^{(3)} = x_2^{(2)} - x_2^{(3)}
\end{aligned} \tag{241}
$$

$$
[J] = \begin{bmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_2}{\partial r_1} \\ \frac{\partial x_1}{\partial r_2} & \frac{\partial x_2}{\partial r_2} \end{bmatrix} = \begin{bmatrix} x_1^{(1)} - x_1^{(3)} & x_2^{(1)} - x_2^{(3)} \\ x_1^{(2)} - x_1^{(3)} & x_2^{(2)} - x_2^{(3)} \end{bmatrix} \tag{242}
$$

- Thus the determinant becomes twice the area of triangle element.

$$
\begin{aligned}
\det[J] &= \frac{\partial x_1}{\partial r_1}\frac{\partial x_2}{\partial r_2} - \frac{\partial x_2}{\partial r_1}\frac{\partial x_1}{\partial r_2} \\
&= (x_1^{(1)} - x_1^{(3)})(x_2^{(2)} - x_2^{(3)}) - (x_2^{(1)} - x_2^{(3)})(x_1^{(2)} - x_1^{(3)})
\end{aligned} \tag{243}
$$

$$
\int_0^1 \int_0^{1-r_1} [B]^T [D] [B] \det[J] dr_2 dr_1 = \det[J] \int_0^1 \int_0^{1-r_1} [B]^T [D] [B] \, dr_2 dr_1 \tag{244}
$$

Drawing the Stiffness Matrix– Triangle 5

• $[B]$ Matrix components are consisted of the series of $\frac{\partial N^{(n)}}{\partial x_i}$, and evaluated by following,

$$\begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial N^{(n)}}{\partial r_1} \\ \frac{\partial N^{(n)}}{\partial r_2} \end{bmatrix} = \frac{1}{\det[J]} \begin{bmatrix} x_2^{(2)} - x_2^{(3)} & -(x_2^{(1)} - x_2^{(3)}) \\ -(x_1^{(2)} - x_1^{(3)}) & x_1^{(1)} - x_1^{(3)} \end{bmatrix} \begin{bmatrix} \frac{\partial N^{(n)}}{\partial r_1} \\ \frac{\partial N^{(n)}}{\partial r_2} \end{bmatrix} \tag{245}$$

•Since $\frac{\partial N^{(n)}}{\partial r_i}$ are all invariables, then automatically $\frac{\partial N^{(n)}}{\partial x_i}$ can be considered as invariables as well,

$$\begin{aligned} \begin{bmatrix} \frac{\partial N^{(1)}}{\partial x_1} & \frac{\partial N^{(2)}}{\partial x_1} & \frac{\partial N^{(3)}}{\partial x_1} \\ \frac{\partial N^{(1)}}{\partial x_2} & \frac{\partial N^{(2)}}{\partial x_2} & \frac{\partial N^{(3)}}{\partial x_2} \end{bmatrix} &= \frac{1}{\det[J]} \begin{bmatrix} x_2^{(2)} - x_2^{(3)} & -(x_2^{(1)} - x_2^{(3)}) \\ -(x_1^{(2)} - x_1^{(3)}) & x_1^{(1)} - x_1^{(3)} \end{bmatrix} \begin{bmatrix} \frac{\partial N^{(1)}}{\partial r_1} & \frac{\partial N^{(2)}}{\partial r_1} & \frac{\partial N^{(3)}}{\partial r_1} \\ \frac{\partial N^{(1)}}{\partial r_2} & \frac{\partial N^{(2)}}{\partial r_2} & \frac{\partial N^{(3)}}{\partial r_2} \end{bmatrix} \\ &= \frac{1}{\det[J]} \begin{bmatrix} x_2^{(2)} - x_2^{(3)} & -(x_2^{(1)} - x_2^{(3)}) \\ -(x_1^{(2)} - x_1^{(3)}) & x_1^{(1)} - x_1^{(3)} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \frac{1}{\det[J]} \begin{bmatrix} x_2^{(2)} - x_2^{(3)} & x_2^{(3)} - x_2^{(1)} & x_2^{(1)} - x_2^{(2)} \\ x_1^{(3)} - x_1^{(2)} & x_1^{(1)} - x_1^{(3)} & x_1^{(2)} - x_1^{(1)} \end{bmatrix} \end{aligned} \tag{246}$$

Drawing the Stiffness Matrix– Triangle 6

- Therefore,$[B]$ matrix components are specifically described as,

$$[B]=\begin{bmatrix}\frac{\partial N^{(1)}}{\partial x_1} & 0 & \frac{\partial N^{(2)}}{\partial x_1} & 0 & \frac{\partial N^{(3)}}{\partial x_1} & 0 \\ 0 & \frac{\partial N^{(1)}}{\partial x_2} & 0 & \frac{\partial N^{(2)}}{\partial x_2} & 0 & \frac{\partial N^{(3)}}{\partial x_2} \\ \frac{\partial N^{(1)}}{\partial x_2} & \frac{\partial N^{(1)}}{\partial x_1} & \frac{\partial N^{(2)}}{\partial x_2} & \frac{\partial N^{(2)}}{\partial x_1} & \frac{\partial N^{(3)}}{\partial x_2} & \frac{\partial N^{(3)}}{\partial x_1}\end{bmatrix}$$

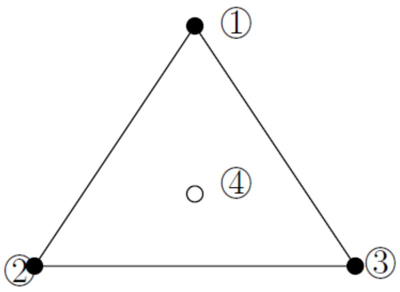
$$=\frac{1}{\det[J]}\begin{bmatrix}x_2^{(2)}-x_2^{(3)} & 0 & x_2^{(3)}-x_2^{(1)} & 0 & x_2^{(1)}-x_2^{(2)} & 0 \\ 0 & x_1^{(3)}-x_1^{(2)} & 0 & x_1^{(1)}-x_1^{(3)} & 0 & x_1^{(2)}-x_1^{(1)} \\ x_1^{(3)}-x_1^{(2)} & x_2^{(2)}-x_2^{(3)} & x_1^{(1)}-x_1^{(3)} & x_2^{(3)}-x_2^{(1)} & x_1^{(2)}-x_1^{(1)} & x_2^{(1)}-x_2^{(2)}\end{bmatrix} \quad (247)$$

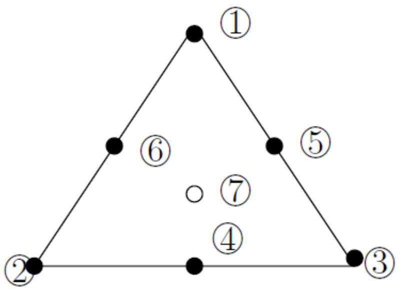
- Moreover, in realizing $[D]$ matrix of being invariables, then automatically integrand all becomes invariables as well. There is no more need for the numerical integrations.

$$\begin{aligned}&\int_0^1\int_0^{1-r_1}[B]^T\,[D]\,[B]\det[J]dr_2dr_1\\ =&\det[J]\int_0^1\int_0^{1-r_1}[B]^T\,[D]\,[B]\,dr_2dr_1\\ =&\det[J][B]^T\,[D]\,[B]\int_0^1\int_0^{1-r_1}dr_2dr_1\\ =&\frac{1}{2}\det[J][B]^T\,[D]\,[B]\end{aligned} \quad (248)$$

Element With Bubble Node

- In using a mixed type formulation such as fluid and incompressibility solid, distinct interpolation functions are used: function not necessary coincide with change and pressure. Proper combination should be determined by inf-sup condition and the experiences.
- For triangle and tetrahedral element, the interpolation functions for the displacement is represented by following.Junctions in the center of force are called bubble nodes.（since the function becomes 0 on the sides of element）.





In specific, interpolation functions for triangles that correspond to this nodes are expressed by,

$$N = 27L_1L_2L_3 \tag{249}$$

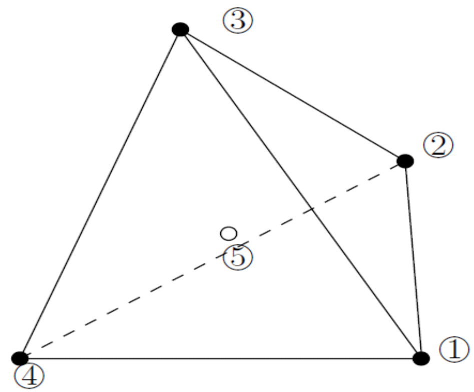
• Introduce the bubble node in 3 nodes.

$$
\begin{aligned}
N^{(1)} &= L_1 \\
N^{(2)} &= L_2 \\
N^{(3)} &= L_3 \\
N^{(4)} &= 27L_1L_2L_3
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\tilde{N}^{(1)} &= N^{(1)} - \tfrac{1}{3}N^{(4)} \\
\tilde{N}^{(2)} &= N^{(2)} - \tfrac{1}{3}N^{(4)} \\
\tilde{N}^{(3)} &= N^{(3)} - \tfrac{1}{3}N^{(4)}
\end{aligned}
\tag{250}
$$

• Introduce the bubble node in 6 nodes.

$$
\begin{aligned}
N^{(1)} &= L_1(2L_1 - 1) \\
N^{(2)} &= L_2(2L_2 - 1) \\
N^{(3)} &= L_3(2L_3 - 1) \\
N^{(4)} &= 4L_2L_3 \\
N^{(5)} &= 4L_3L_1 \\
N^{(6)} &= 4L_1L_2 \\
N^{(7)} &= 27L_1L_2L_3 \quad \text{(bubble.)}
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\tilde{N}^{(1)} &= N^{(1)} + \tfrac{1}{9}N^{(7)} \\
\tilde{N}^{(2)} &= N^{(2)} + \tfrac{1}{9}N^{(7)} \\
\tilde{N}^{(3)} &= N^{(3)} + \tfrac{1}{9}N^{(7)} \\
\tilde{N}^{(4)} &= N^{(4)} + \tfrac{1}{9}N^{(7)} \\
\tilde{N}^{(5)} &= N^{(5)} + \tfrac{1}{9}N^{(7)} \\
\tilde{N}^{(6)} &= N^{(6)} + \tfrac{1}{9}N^{(7)}
\end{aligned}
\tag{251}
$$

Tetrahedron With Bubble Node



- In tetrahedral, the interpolation functions for the bubble node in the center of mass of an element may be,

$$N_5 = 256 L_1 L_2 L_3 L_4 \tag{252}$$

- In 4 nodes: Modify $N_1 \sim N_4$ in order for the center of mass $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ to be 0.

$$\tilde{N}^{(1)} = N^{(1)} - \frac{1}{4} N^{(5)} \tag{253}$$

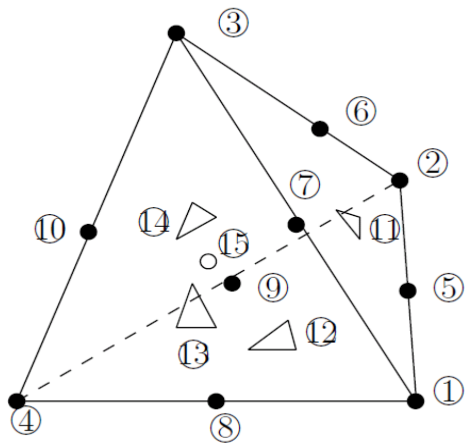
$$\tilde{N}^{(2)} = N^{(2)} - \frac{1}{4} N^{(5)} \tag{254}$$

$$\tilde{N}^{(3)} = N^{(3)} - \frac{1}{4} N^{(5)} \tag{255}$$

$$\tilde{N}^{(4)} = N^{(4)} - \frac{1}{4} N^{(5)} \tag{256}$$

(257)

• In a case of 10 nodal points, bubble the total 5 points at the center of force in each plane and element.



Bubble up the center of force at each plane for 10 nodes.

$$1\text{の対面}\quad 14 \quad \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \tag{258}$$

$$2\text{の対面}\quad 13 \quad \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \tag{259}$$

$$3\text{の対面}\quad 12 \quad \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix} \tag{260}$$

$$4\text{の対面}\quad 11 \quad \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \tag{261}$$

$$\tag{262}$$

$$N_{11} = 27L_1L_2L_3 \tag{263}$$

$$N_{12} = 27L_1L_2L_4 \tag{264}$$

$$N_{13} = 27L_1L_3L_4 \tag{265}$$

$$N_{14} = 27L_2L_3L_4 \tag{266}$$

$$\tag{267}$$

Modify $N^{(1)} \sim N^{(10)}$ for now.

$$N^{(1)} = N^{(1)} + \frac{1}{9}(N_{11} + N_{12} + N_{13}) \tag{268}$$

$$N^{(2)} = N^{(2)} + \frac{1}{9}(N_{11} + N_{12} + N_{14}) \tag{269}$$

$$N^{(3)} = N^{(3)} + \frac{1}{9}(N_{11} + N_{13} + N_{14}) \tag{270}$$

$$N^{(4)} = N^{(4)} + \frac{1}{9}(N_{12} + N_{13} + N_{14}) \tag{271}$$

$$N^{(5)} = N^{(5)} - \frac{4}{9}(N_{11} + N_{12}) \tag{272}$$

$$N^{(6)} = N^{(6)} - \frac{4}{9}(N_{11} + N_{14}) \tag{273}$$

$$N^{(7)} = N^{(7)} - \frac{4}{9}(N_{11} + N_{13}) \tag{274}$$

$$N^{(8)} = N^{(8)} - \frac{4}{9}(N_{12} + N_{13}) \tag{275}$$

$$N^{(9)} = N^{(9)} - \frac{4}{9}(N_{12} + N_{14}) \tag{276}$$

$$N^{(10)} = N^{(10)} - \frac{4}{9}(N_{13} + N_{14}) \tag{277}$$

- Bubblc $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ the element of the center of mass,

$$N^{(15)} = 256 L_1 L_2 L_3 L_4 \tag{278}$$

- Modify $N^{(1)} \sim N^{(14)}$.

$$N^{(1)} \sim N^{(14)} = N^{(1)} \sim N^{(4)} - \frac{1}{64} N^{(15)} \tag{279}$$

$$N^{(5)} \sim N^{(10)} = N^{(5)} \sim N^{(10)} + \frac{1}{8} N^{(15)} \tag{280}$$

$$N^{(11)} \sim N^{(14)} = N^{(11)} \sim N^{(14)} - \frac{27}{64} N^{(15)} \tag{281}$$

2004 Advanced Nonlinear Finite Element Method Exercises 3

- Verify that the element matrix becomes (30) when x and u are discretized respectively as we can see in (21)and (23).
- In equally taking the nodal points in single dimension, verify (26)and (27).
- Verify (57) through(39).
- Verify (60) from(58).
- From the basics to the Gauss integration(67),verify (71). Also, find the weight and the coordinates of sampling points when n=3.
- Start with 4 noded quadrilateral shape function, increase the number of nodes to obtain the shape function for 9 nodes. Verify from (133) to (161). Verify that from (154) to(161), and from (112)to (119) coincide with each other.
- The correspondence of area coordinates and natural coordinates for triangle is expressed by(173) to(175). Now, verify if this relationship can be established.
- Complete the attached programming exercises.