PartV

Interaction of Light and Matter

11 Classical Electromagnetic Field

In this section, we discuss the classical electromagnetic field that obeys the Maxwell 's equation to help us understand the quantum phenomena associated with light.

11.1 Maxwell 's Equation

To begin, let us consider a case with a particle in vacuum, which carries an electric charge e_i, and can be found in the coordinates $\vec{r}_i$. The Maxwell 's model for $i = 1, \cdots, N$ becomes

$$\begin{aligned} \mathrm{rot}\,\vec{E} + \frac{\partial \vec{B}}{\partial \vec{t}} &= 0 \\ \mathrm{rot}\,\vec{H} - \frac{\partial \vec{D}}{\partial t} &= \vec{j} \\ \mathrm{div}\,\vec{D} &= \rho \\ \mathrm{div}\,\vec{B} &= 0 \end{aligned}$$

The vacuum permittivity and permeability are used to write in the form:

$$\begin{aligned} \vec{D} &= \epsilon_0 \vec{E} \\ \vec{H} &= \frac{1}{\mu_0} \vec{B} \end{aligned}$$

For the charge density and the current density, the coordinates of the particle is used and they are written as:

$$\begin{aligned} \rho(\vec{r}) &= \sum_{i=1}^{N} e_i \delta(\vec{r} - \vec{r}_i) \\ \vec{j}(\vec{r}) &= \sum_{i=1}^{N} e_i \dot{\vec{r}}_i \delta(\vec{r} - \vec{r}_i) \end{aligned}$$

Note that the equations satisfy the conservation of electric charge: [207]

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, \vec{j} = 0$$

As for another fundamental equation to this system, we consider an equation of motion for a particle in $\vec{r}_i$ that obeys the Lorentz force. Here, we let m_i be the particle mass:

$$m_i \ddot{\vec{r}}_i = e_i \vec{E}(\vec{r}_i) + e_i \dot{\vec{r}}_i \times \vec{B}(\vec{r}_i)$$

The time resolution for the particle 's kinetic energy T is expressed by [208]

$$\dot{T} = \int dV \vec{E} \cdot \vec{j}$$

Here we assume V to be an arbitrary region that includes $\vec{r}_i$. The Maxwell 's

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$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \sum_i e_i \frac{\partial}{\partial t} \delta(\vec{r} - \vec{r}_i(t)) \\ &= \sum_i e_i \dot{\vec{r}}_i \cdot \vec{\nabla}_{r_i} \delta(\vec{r} - \vec{r}_i(t)) \\ &= \sum_i e_i \dot{\vec{r}}_i \cdot (-1) \vec{\nabla}_r \delta(\vec{r} - \vec{r}_i(t)) \end{aligned}$$

Further,

$$\mathrm{div}\, \vec{j} = \sum_i e_i \dot{\vec{r}}_i \cdot \vec{\nabla}_r \delta(\vec{r} - \vec{r}_i(t))$$

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$$\begin{aligned} \frac{d}{dt} \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2 &= \sum_i m_i \dot{\vec{r}}_i \cdot \ddot{\vec{r}}_i = \sum_i e_i \dot{\vec{r}}_i \cdot (\vec{E}_i + \dot{\vec{r}}_i \times \vec{B}_i) \\ &= \sum_i e_i \dot{\vec{r}}_i \cdot \vec{E}_i = \int dV \vec{E} \cdot \vec{j} \\ & E_i = E(\vec{r}_i), \quad B_i = B(\vec{r}_i) \end{aligned}$$

equation provides [209]

$$\begin{aligned}\vec{P} &= \vec{E}\times\vec{H}\\ E_{em} &= \int dV\,\mathcal{E}_{em}\\ \mathcal{E}_{em} &= \frac{1}{2}(\epsilon_0\vec{E}^2+\mu_0\vec{H}^2)\end{aligned}$$

Hence,

$$\frac{d}{dt}(T+E_{em})+\int_{\partial V} d\vec{S}\cdot\vec{P}=0$$

We understand that P denotes the momentum of the electromagnetic field while E_{em} denotes the energy of the electromagnetic field. (P is known as the Poynting vector.)

11.2 The Vector Potential and Scalar Potential

First, note $\operatorname{div}\vec{B}=0$ can give
[210]

[209] Maxwell 's equation can give

$$\begin{aligned}\vec{H}\cdot\operatorname{rot}\vec{E}+\mu_0\vec{H}\cdot\dot{\vec{H}} &= 0\\ \vec{E}\cdot\operatorname{rot}\vec{H}-\epsilon_0\vec{E}\cdot\dot{\vec{E}} &= \vec{E}\cdot\vec{j}\end{aligned}$$

We take the difference between the equations above:

$$-\operatorname{div}(\vec{E}\times\vec{H})-\frac{1}{2}\frac{d}{dt}(\epsilon_0\vec{E}^2+\mu_0\vec{H}^2)=\vec{E}\cdot\vec{j}$$

$$\operatorname{div}\vec{P}+\frac{d\mathcal{H}_{em}}{dt}+\vec{E}\cdot\vec{j}=0$$

Thus,

$$\begin{aligned}\operatorname{div}(\vec{A}\times\vec{B}) &= \partial_i\epsilon_{ijk}A_jB_k\\ &= \epsilon_{ijk}(\partial_iA_j)B_k+\epsilon_{ijk}A_j(\partial_i)B_k\\ &= \epsilon_{kij}(\partial_iA_j)B_k-\epsilon_{jik}A_j(\partial_i)B_k\\ &= \operatorname{rot}\vec{A}\cdot\vec{B}-\vec{A}\cdot\operatorname{rot}\vec{B}\end{aligned}$$

[210] An arbitrary vector field $\vec{X}$ can be expressed by

$$\begin{aligned}\vec{X}=\vec{X}_T+\vec{X}_L\\ \operatorname{div}\vec{X}_T=0\\ \operatorname{rot}\vec{X}_L=0\end{aligned}$$

Note that $\vec{X}_L$ and $\vec{X}_T$ are respectively called the longitudinal and transverse components. When the vector field described above is definable in all region of space, we may express the field by the potential:

$$\begin{aligned}\vec{X}_T &= \mathrm{rot}\,\vec{A}\\ \vec{X}_L &= \mathrm{grad}\,\phi\end{aligned}$$

[211] [212]

$$\vec{B} = \mathrm{rot}\,\vec{A}$$

[211] The Fourier expansion for the arbitrary field is written as

$$\vec{X}(\vec{r}) = \sum_{\vec{k}} X_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$$

which yields

$$\begin{aligned} \mathrm{div}\,X &= \sum_{\vec{k}} i\vec{k}\cdot\vec{X}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \\ \mathrm{rot}\,X &= \sum_{\vec{k}} i\vec{k}\times\vec{X}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \end{aligned}$$

Now, let us write down the orthonormalization of the right-handed system for

$$\vec{e}_{\vec{k}\sigma=0} = \frac{\vec{k}}{k}, \quad \vec{e}_{\vec{k}\sigma=1}, \quad \vec{e}_{\vec{k}\sigma=2}$$

we obtain

$$\begin{aligned} \vec{X}_L &= \sum_{\vec{k}} (\vec{X}_{\vec{k}}\cdot\vec{e}_{\vec{k},0})\vec{e}_{\vec{k},0} e^{i\vec{k}\cdot\vec{r}} = \sum_{\vec{k}} \frac{(\vec{X}_{\vec{k}}\cdot\vec{k})\vec{k}}{k^2} e^{i\vec{k}\cdot\vec{r}} \\ (\vec{X}_L)_\alpha &= \sum_{\vec{k}} \frac{k_\alpha k_\beta}{k^2} X_\beta e^{i\vec{k}\cdot\vec{r}} \\ \vec{X}_T &= \sum_{\vec{k}} \sum_{\sigma=1,2} (\vec{X}_{\vec{k}}\cdot\vec{e}_{\vec{k}\sigma})\vec{e}_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}} = \sum_{\vec{k}} \left(\vec{X}_{\vec{k}} - \frac{(\vec{X}_{\vec{k}}\cdot\vec{k})\vec{k}}{k^2}\right) e^{i\vec{k}\cdot\vec{r}} \\ (\vec{X}_T)_\alpha &= \sum_{\vec{k}} (\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}) X_\beta e^{i\vec{k}\cdot\vec{r}} = \sum_{\vec{k}} (\sum_{\sigma=1,2} (\vec{e}_{\vec{k}\sigma})_\alpha (\vec{e}_{\vec{k}\sigma})_\beta) X_\beta e^{i\vec{k}\cdot\vec{r}} \end{aligned}$$

Conditions for the complete system give

$$\sum_\sigma (\vec{e}_{\vec{k}\sigma})_\alpha (\vec{e}_{\vec{k}\sigma})_\beta = \frac{k_\alpha k_\beta}{k^2} + \sum_{\sigma=1,2} (\vec{e}_{\vec{k}\sigma})_\alpha (\vec{e}_{\vec{k}\sigma})_\beta = \delta_{\alpha\beta}$$

such that we write

$$\sum_{\sigma=1,2} (\vec{e}_{\vec{k}\sigma})_\alpha (\vec{e}_{\vec{k}\sigma})_\beta = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$$

This is valid since

$$\begin{aligned} \vec{v} &= (\vec{v}\cdot\vec{e}_\sigma)\vec{e}_\sigma \\ v_\alpha &= v_\beta (\vec{e}_\sigma)_\beta (\vec{e}_\sigma)_\alpha \end{aligned}$$

is written for the arbitrary vector $\vec{v}$. We can further write the above as

$$(\vec{e}_\sigma)_\beta (\vec{e}_\sigma)_\alpha = \delta_{\alpha\beta}$$

A similar formula to the expansion of the function is given as

$$\sum_j \psi_j^*(x)\psi_j(x') = \delta(x - x')$$

Thus,

$$\mathrm{rot}\left(\vec{E}+\frac{\partial \vec{A}}{\partial t}\right)=0$$

Rewrite the equation above of the physical quantity:

$$\vec{E}=-\frac{\partial \vec{A}}{\partial t}-\vec{\nabla}\phi$$

[212] Let us summarize the relationship between differential-form and the vector calculus formulas:

$$\begin{aligned}
\Omega_0 &= f \\
d\Omega_0 &= \partial_i f\ dx_i \quad : \mathrm{grad}\, f \\
d^2\Omega_0 &= \partial_j \partial_i f\ dx_j \wedge dx_i = 0 \quad : \mathrm{rot}\,\mathrm{grad}\, f = 0 \\
\Omega_1 &= A_i dx_i \quad : \vec{A} \\
d\Omega_1 &= \partial_j A_i\ dx_j \wedge dx_i \quad : \mathrm{rot}\, \vec{A} \\
d^2\Omega_1 &= \partial_k \partial_j A_i\ dx_k \wedge dx_j \wedge dx_i = 0 \quad : \mathrm{div}\,\mathrm{rot}\, \vec{A} = 0 \\
\Omega_2 &= A_i * dx_i = \epsilon_{ijk} A_i dx_j \wedge dx_k \quad : \vec{A} \\
d\Omega_2 &= \partial_\ell A_i dx_\ell * dx_i = \partial_i A_i\ dx_1 \wedge dx_2 \wedge dx_3 \quad : \mathrm{div}\, \vec{A} \\
d^2\Omega_2 &= 0
\end{aligned}$$

Here we define

$$\begin{aligned}
*1 &= dx_1 \wedge dx_2 \wedge dx_3 \\
*dx_1 &= dx_2 \wedge dx_3,\ *dx_2 = dx_3 \wedge dx_1,\ *dx_3 = dx_1 \wedge dx_2, \\
*(dx_1 \wedge dx_2) &= dx_3,\quad *(dx_2 \wedge dx_3) = dx_1,\quad *(dx_3 \wedge dx_1) = dx_2, \\
*(dx_1 \wedge dx_2 \wedge dx_3) &= 1
\end{aligned}$$

giving

$$\begin{aligned}
A &= A_i dx_i \\
*dA &= \mathrm{rot}\, A = (\mathrm{rot}\, A)_i dx_i \\
*d*A &= \mathrm{div}\, A \\
d\phi &= \mathrm{grad}\, \phi = \nabla\phi \\
*d*d\phi &= \Delta\phi
\end{aligned}$$

$$\begin{aligned}
\mathrm{div}\,\mathrm{rot}\, \vec{A} &= *d*(*dA) = d(dA) = 0 \\
\mathrm{rot}\,\mathrm{grad}\, f &= *d(df) = 0
\end{aligned}$$

For the integral formula, we can write

$$\begin{aligned}
\int_V d\Omega_2 &= \int_{\partial V} \Omega_2 \quad : \quad \int_V \mathrm{div}\, \vec{A}\, dV = \int_{\partial V} \vec{A}\cdot d\vec{S} \\
\int_S d\Omega_1 &= \int_{\partial S} \Omega_1 \quad : \quad \int_S \mathrm{rot}\, \vec{A}\cdot d\vec{S} = \int_{\partial S} \vec{A}\cdot d\vec{r} \\
\int_L d\Omega_0 &= \int_{\partial L} \Omega_0 \quad : \quad \int_L \mathrm{grad}\, f\cdot d\vec{r} = f(\vec{r})\Big|_{\vec{r}=\vec{r}_{ini}}^{\vec{r}=\vec{r}_{fin}}
\end{aligned}$$

by using the vector potential $\vec{A}$ and the scalar potential ϕ.

Note that the physical quantities $\vec{E}$ and $\vec{B}$ stay constant even though the gauge transformation is conducted under the arbitrary space-time equation $\chi(\vec{r}, t)$:

$$\begin{aligned} \vec{A} &\to \vec{A}' = \vec{A} + \vec{\nabla}\chi \\ \phi &\to \phi' = \phi - \frac{\partial \chi}{\partial t} \end{aligned}$$

$$\vec{E}' = \vec{E}, \quad \vec{B}' = \vec{B}$$

We must note that there are certain degrees of freedom left in the potential expression. The Maxwell 's equation is rewritten by using such potential. rot $\vec{H} - \dot{\vec{D}} = \vec{j}$ gives
[213]

$$\begin{aligned} -\Box \vec{A} &\equiv \frac{1}{c^2}\ddot{\vec{A}} - \Delta \vec{A} = -\vec{\nabla}(\text{div}\,\vec{A} + \frac{1}{c^2}\dot{\phi}) + \mu_0 \vec{j} \\ c^2 &= \frac{1}{\epsilon_0 \mu_0} \end{aligned}$$

While div $\vec{D} = \rho$ gives

$$-\Delta\phi = \text{div}\,\dot{\vec{A}} + \frac{1}{\epsilon_0}\rho$$

Let us have a particular Coulomb gauge

$$\text{div}\,\vec{A} = 0$$

and by which we obtain two relational expressions of the Maxwell 's equation:

$$\begin{aligned} -\Box \vec{A} &= = \mu_0 \vec{J} \\ -\Delta\phi &= \frac{1}{\epsilon_0}\rho \end{aligned}$$

Note that the equation for the scalar potential

$$\vec{J} = \vec{j} - \epsilon_0 \vec{\nabla}\dot{\phi}$$

can be easily integrated: [214]

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{e_i}{|\vec{r} - \vec{r}_i|}$$

[213]The equation $\frac{1}{\mu_0}\text{rot}\,\text{rot}\,\vec{A} - \epsilon_0(-\ddot{\vec{A}} - \vec{\nabla}\phi) = \vec{j}$ gives $\vec{\nabla}\text{div}\,\vec{A} - \Delta\vec{A} + \frac{1}{c^2}\left(\ddot{\vec{A}} + \nabla\dot{\phi}\right) = \mu_0\vec{j}$

[214]The solution for

$$-\Delta f(\vec{r}) = \delta(\vec{r})$$

is

$$f(\vec{r}) = \frac{1}{4\pi r}$$

Hence, we can further rewrite the first equation into the form:

$$\vec{J} = \sum_i \Big(-\frac{\partial}{\partial t}\vec{\nabla}\frac{e_i}{4\pi|\vec{r}-\vec{r}_i|} + e_i\,\dot{\vec{r}}_i\delta(\vec{r}-\vec{r}_i)\Big)$$

Note that [215]

$$\mathrm{div}\,\vec{J} = 0$$

Let us now suppose that the system is in a box with the volume V and the edge length of L. We conduct the Fourier transformation of A under the periodic boundary condition:

[216]

$$\begin{aligned}\vec{A} &= \frac{1}{\sqrt{V}}\sum_{\vec{k}}\vec{A}_{\vec{k}}e^{i\vec{k}\cdot\vec{r}} \\ \vec{k} &= \frac{2\pi}{L}(n_x,n_y,n_z), \quad n_i = \cdots, -2,-1,0,1,2,\cdots\end{aligned}$$

The vector potential can be written in the following form by using $\vec{k}\cdot\vec{A}_{\vec{k}} = 0$ which is given by $\mathrm{div}\,\vec{A} = 0$:

[217]

$$\hat{k}\cdot\vec{e}_{k\sigma=1} = 0, \hat{k}\cdot\vec{e}_{k\sigma=2} = 0, \quad \vec{e}_{k1}\cdot\vec{e}_{k2} = 0$$

giving

$$\vec{A}(\vec{r},t) = \frac{1}{\sqrt{\epsilon_0 V}}\sum_{\vec{k}}\sum_{\sigma=1,2}\vec{e}_{\vec{k}\sigma}q_{\vec{k}\sigma}(t)e^{i\vec{k}\cdot\vec{r}}$$

We can use the fact that A is being real to express $\vec{e}_{-\vec{k}\sigma} = \vec{e}_{\vec{k}\sigma}$. This allows us to use $\vec{A}_{-\vec{k}} = \vec{A}^*_{\vec{k}}$. Therefore,

$$q^*_{\vec{k}\sigma}(t) = q_{-\vec{k}\sigma}(t)$$

In the same way, we can write

$$\phi(\vec{r},t) = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\phi_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{r}}$$

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$$\mathrm{div}\,\vec{J} = -\frac{\partial}{\partial t}\epsilon_0\Delta\phi + \mathrm{div}\,\vec{j} = \frac{\partial}{\partial t}\rho + \mathrm{div}\,\vec{j} = 0$$

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$$A_{\vec{k}} = \frac{1}{\sqrt{V}}\int dV\vec{A}(\vec{r})e^{-i\vec{k}\cdot\vec{r}}$$

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$$\vec{A}_{\vec{k}} = \frac{1}{\sqrt{\epsilon_0}}\sum_{\sigma=1,2}\vec{e}_{\vec{k}\sigma}q_{\vec{k}\sigma}(t)$$

$$\vec{j}(\vec{r},t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \vec{j}_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{r}}$$

Given the information above, we now move on to discuss $\Box\vec{A} = \mu_0\vec{J}$. The longitudinal components (components in $\vec{k}$ direction) can be expressed by using div $\vec{A} = 0$:

$$\epsilon_0 i k^2 \dot{\phi}_{\vec{k}} \quad - \quad \vec{k}\cdot\vec{j}_{\vec{k}} = 0$$

By time-differentiating the Poisson 's equation, and further using the continuity equation, we write:

$$\epsilon_0 \Delta\dot{\phi} = -\dot{\rho} = \nabla_r \cdot \vec{j}$$

And from which, the Fourier components are written as $-k^2\dot{\phi}_k = i\vec{k}\cdot\vec{j}_{\vec{k}}$ and therefore, the relational expression for the longitudinal components is automatically satisfied. Now, for the transverse components: [218]

$$\begin{aligned}
\ddot{q}_{\vec{k}\sigma} + \omega_{\vec{k}}^2 q_{\vec{k}\sigma} &= \frac{1}{\sqrt{\epsilon_0 V}} \vec{e}_{\vec{k}\sigma} \cdot \int dV \vec{j}(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} \\
&= \frac{1}{\sqrt{\epsilon_0 V}} \sum_i e_i \,(\vec{e}_{\vec{k}\sigma}\cdot\dot{\vec{r}}_i) e^{-i\vec{k}\cdot\vec{r}_i} \qquad (\omega = ck)
\end{aligned}$$

This is the equation the vector potential must satisfy, and which is in fact equivalent to the Maxwell 's equation. The equation describes the forced oscillation for each polarized light $\vec{e}_{\vec{k}\sigma}$.

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$$\begin{aligned}
\vec{e}_{\vec{k}\sigma} \cdot \Big(-\Box\vec{A}(\vec{r}) \Big) &= \vec{e}_{\vec{k}\sigma} \cdot \frac{1}{\sqrt{\epsilon_0 V}} \sum_{\vec{k}} \sum_{\sigma=1,2} \vec{e}_{\vec{k}\sigma} \big(\frac{1}{c^2}\ddot{q}_{\vec{k}\sigma} + k^2 q_{\vec{k}\sigma}\big) e^{i\vec{k}\cdot\vec{r}} = \frac{1}{\sqrt{\epsilon_0 V}} \sum_{\vec{k}} \big(\frac{1}{c^2}\ddot{q}_{\vec{k}\sigma} + k^2 q_{\vec{k}\sigma}\big) e^{i\vec{k}\cdot\vec{r}} \\
\vec{e}_{\vec{k}\sigma} \cdot \mu_0 \vec{J}(\vec{r}) &= \vec{e}_{\vec{k}\sigma} \cdot \vec{j}(\vec{r}) = \mu_0 \frac{1}{\sqrt{V}} \, \vec{e}_{\vec{k}\sigma} \cdot \sum_{\vec{k}} \vec{j}_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}
\end{aligned}$$

Thus,

$$\frac{1}{c^2}\ddot{q}_{\vec{k}\sigma} + k^2 q_{\vec{k}\sigma} = \mu_0\sqrt{\epsilon_0}\,\vec{e}_{\vec{k}\sigma}\vec{j}_{\vec{k}} = \frac{\mu_0\sqrt{\epsilon_0}}{\sqrt{V}} \int dV \vec{e}_{\vec{k}\sigma}\vec{j}(\vec{r}) e^{-i\vec{k}\cdot\vec{r}}$$

11.3 Classical Field Equations

First, we consider the energy of the electromagnetic field by dividing it into two parts: [219] [220]

$$\begin{aligned}
E_{em} &= \frac{1}{2}\int dV\left(\epsilon_0(\dot{\vec{A}}+\vec{\nabla}\phi)^2+\frac{1}{\mu_0}(\mathrm{rot}\,\vec{A})^2\right) \\
&= E_{rad}+E_{coulomb} \\
E_{rad} &= \frac{1}{2}\int dV\left(\epsilon_0\dot{\vec{A}}^2+\frac{1}{\mu_0}(\mathrm{rot}\,\vec{A})^2\right) \\
E_{coulomb} &= \epsilon_0\frac{1}{2}\int dV\left(2\dot{\vec{A}}\vec{\nabla}\phi+\vec{\nabla}\phi\cdot\vec{\nabla}\phi\right) \\
&= -\epsilon_0\frac{1}{2}\int dV\left(2\phi\;\mathrm{div}\,\dot{\vec{A}}+\phi\Delta\phi\right) \\
&= \frac{1}{2}\int dV\;\rho\phi \\
&= \frac{1}{2}\sum_{ij}\frac{e_ie_j}{4\pi\epsilon_0|\vec{r}_i-\vec{r}_j|} \\
&= \sum_{i<j}\frac{e_ie_j}{4\pi\epsilon_0|\vec{r}_i-\vec{r}_j|}+(expansiontermsoftheself-interaction)
\end{aligned}$$

Note that $E_{coulomb}$ is the Coulomb interaction (we do not consider the expansion terms of self-interaction here) while E_{rad} is the energy of radiation field. If we have

$$p_{\vec{k}\sigma}(t)=\dot{q}_{-\vec{k}\sigma}(t)$$

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$$\int dV\mathrm{div}\,(f\vec{\nabla}g)=\int dV\vec{\nabla}f\cdot\vec{\nabla}g+\int dV f\Delta g=\int_{\partial V}d\vec{S}\cdot f\vec{\nabla}g$$

The boundary terms are cancelled due to the periodic boundary condition thus,

$$\int dV\vec{\nabla}f\cdot\vec{\nabla}g=-\int dV f\Delta g=-\int dV(\Delta f)g$$

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$$\int \mathrm{div}\,(\phi\dot{\vec{A}})=\int_{\partial V}d\vec{S}\cdot\phi\dot{\vec{A}}=0$$

Thus,

$$\int dV\phi\mathrm{div}\,\dot{\vec{A}}=-\int dV\dot{\vec{A}}\cdot\vec{\nabla}\phi$$

then we can write

$$\begin{aligned}\vec{A}(\vec{r},t) &= \frac{1}{\sqrt{\epsilon_0 V}}\sum_{\vec{k}}\sum_{\sigma=1,2}\vec{e}_{\vec{k}\sigma}q_{\vec{k}\sigma}(t)e^{i\vec{k}\cdot\vec{r}} \\ \dot{\vec{A}}(\vec{r},t) &= \frac{1}{\sqrt{\epsilon_0 V}}\sum_{\vec{k}}\sum_{\sigma=1,2}\vec{e}_{\vec{k}\sigma}p_{\vec{k}\sigma}(t)e^{-i\vec{k}\cdot\vec{r}}\end{aligned}$$

We substitute the above into E_{rad}: [221]

$$E_{rad} = \frac{1}{2}\sum_{\vec{k}}\sum_{\sigma=1,2}\left(p_{\vec{k}\sigma}p_{-\vec{k}\sigma} + c^2k^2 q_{\vec{k}\sigma}q_{-\vec{k}\sigma}\right)$$

By adding the kinetic energy $T = \frac{1}{2}\sum_i \ddot{\vec{r}}^2$, the classical energy is expressed in the form:

$$H = T + E_{rad} + E_{coulomb}$$

We let $q_{\vec{k}\sigma}, p_{\vec{k}\sigma}$, of radiation field and $\vec{r}_i$ of the particle system be the canonical variables, while we let whose conjugate momenta be

$$\vec{P}_i = m_i\dot{\vec{r}}_i + e_i\vec{A}(\vec{r}_i) = m_i\dot{\vec{r}}_i + e_i\vec{A}_i$$

The Hamiltonian is therefore given by:

$$\begin{aligned}H &= H_{part} + H_{rad} + H_{coulomb} \\ H_{part} &= \sum_i \frac{1}{2m_i}(\vec{P}_i - e_i\vec{A}(\vec{r}_i))^2 \\ &= \sum_i \frac{1}{2m_i}\left(\vec{P}_i - e_i\frac{1}{\sqrt{\epsilon_0 V}}\sum_{\vec{k}\sigma}\vec{e}_{\vec{k}\sigma}q_{\vec{k}\sigma}e^{i\vec{k}\cdot\vec{r}_i}\right)^2 \\ H_{rad} &= +\frac{1}{2}\sum_{\vec{k}}\sum_{\sigma=1,2}\left(p_{\vec{k}\sigma}p_{-\vec{k}\sigma} + c^2k^2 q_{\vec{k}\sigma}q_{-\vec{k}\sigma}\right) \\ H_{coulomb} &= \sum_{i<j}\frac{e_ie_j}{4\pi\epsilon_0|\vec{r}_i - \vec{r}_j|}\end{aligned}$$

[221]For the energy of a magnetic field:

$$\begin{aligned}\mathrm{div}\,(\vec{A}\times\mathrm{rot}\,\vec{A}) &= \mathrm{rot}\,\vec{A}\cdot\mathrm{rot}\,\vec{A} - \vec{A}\cdot\mathrm{rot}\,\mathrm{rot}\,\vec{A}, \quad (\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{\nabla}\times\vec{A}\cdot\vec{B} - \vec{A}\cdot\vec{\nabla}\times\vec{B}) \\ &= \mathrm{rot}\,\vec{A}\cdot\mathrm{rot}\,\vec{A} - \vec{A}\cdot\mathrm{grad}\,\mathrm{div}\,A + \vec{A}\cdot\Delta\vec{A}\end{aligned}$$

which cancels the surface terms thus using $\mathrm{div}\,\vec{A} = 0$, we can write

$$\int dV\mathrm{rot}\,\vec{A}\cdot\mathrm{rot}\,\vec{A} = -\int dV\vec{A}\cdot\Delta\vec{A}$$

The canonical equations are given by:

$$\begin{aligned}
\frac{\partial H}{\partial q_{\vec{k}\sigma}} &= -\dot{p}_{\vec{k}\sigma} \\
\frac{\partial H}{\partial p_{\vec{k}\sigma}} &= \dot{q}_{\vec{k}\sigma} \\
\frac{\partial H}{\partial r^{\alpha}_{\vec{k}\sigma}} &= -\dot{P}^{\alpha}_{\vec{k}\sigma} \\
\frac{\partial H}{\partial P^{\alpha}_{\vec{k}\sigma}} &= \dot{r}^{\alpha}_{\vec{k}\sigma}
\end{aligned}$$

Therefore, the equation of motion for the particles is written as [222]

$$m_i \ddot{\vec{r}}_i = e_i(\vec{E}(\vec{r}_i) + \dot{\vec{r}}_i \times B(\vec{r}_i))$$

The Maxwell 's equation is also written as

[222] For the particle system we can write

$$\begin{aligned} \dot{r}_i^\alpha &= \frac{\partial H}{\partial P_i^\alpha} \\ &= \frac{1}{m_i}(P_i^\alpha - e_i A^\alpha(\vec{r}_i)) \\ -\dot{P}_i^\alpha &= \frac{\partial H}{\partial r_i^\alpha} \\ &= \frac{1}{m_i}(\vec{P}_i - e_i \vec{A}(\vec{r}_i)) \cdot (-e_i)\partial_\alpha \vec{A}(\vec{r}_i) + e_i \partial_\alpha \phi(\vec{r}_i) \\ &= -e_i \dot{r}_i^\beta \partial_\alpha A^\beta(\vec{r}_i) + e_i \partial_\alpha \phi(\vec{r}_i) \end{aligned}$$

Here note:

$$\begin{aligned} \frac{\partial}{\partial r_i^\alpha} H_{coulomb} &= \frac{\partial}{\partial r_i^\alpha} \frac{1}{4\pi\epsilon_0} \sum_{a<b} \frac{1}{|\vec{r}_a - \vec{r}_b|} \\ &= \frac{\partial}{\partial r_i^\alpha} \frac{1}{4\pi\epsilon_0} \sum_{j(\neq i)} \frac{1}{|\vec{r}_a - \vec{r}_b|} \\ &= \partial_\alpha e_i \phi(\vec{r}_i) = \partial_\alpha e_i \phi_i \end{aligned}$$

$$\begin{aligned} \vec{A}_i &= \vec{A}(\vec{r}_i) \\ \frac{d}{dt}\vec{A}_i &= \left.\frac{d\vec{A}(\vec{r})}{dt}\right|_{\vec{r}=\vec{r}_i} + \dot{\vec{r}_i} \cdot \vec{\nabla}_{\vec{r}_i} \vec{A}_i \end{aligned}$$

Hence,

$$\begin{aligned} m_i \ddot{r}_i^\alpha &= \dot{P}_i^\alpha - e_i \dot{A}_i^\alpha(\vec{r}_i) - e_i \dot{\vec{r}_i} \cdot \vec{\nabla}_i A_i^\alpha(\vec{r}_i) \\ &= e_i \dot{r}_i^\beta \partial_\alpha A^\beta(\vec{r}_i) - e_i \partial_\alpha \phi(\vec{r}_i) \\ &\quad - e_i \dot{A}_i^\alpha(\vec{r}_i) - e_i \dot{r}_i^\beta \partial_\beta A_i^\alpha(\vec{r}_i) \\ &= e_i\big(- \partial_\alpha \phi(\vec{r}_i) - \dot{A}_i^\alpha(\vec{r}_i) + \dot{r}_i^\beta \partial_\alpha A^\beta(\vec{r}_i) - \dot{r}_i^\beta \partial_\beta A_i^\alpha(\vec{r}_i)\big) \end{aligned}$$

Further,

$$\begin{aligned} (\dot{\vec{r}} \times \mathrm{rot}\, \vec{A})_\alpha &= \epsilon_{\alpha\beta\gamma} \dot{r}^\beta \epsilon_{\gamma\eta\xi} \partial_\eta A^\xi \\ &= (\delta_{\alpha\eta}\delta_{\beta\xi} - \delta_{\alpha\xi}\delta_{\beta\eta}) \dot{r}^\beta \partial_\eta A^x i \\ &= \dot{r}^\beta \partial_\alpha A^\beta - \dot{r}^\beta \partial_\beta A^\alpha \end{aligned}$$

Thus,

$$m_i \ddot{\vec{r}}_i = e_i(\vec{E}(\vec{r}_i) + \dot{\vec{r}}_i \times B(\vec{r}_i))$$

[223]

$$\ddot{q}_{-\vec{k}\sigma} + c^2k^2 q_{-\vec{k}\sigma} = \frac{1}{\sqrt{\epsilon_0 V}} \sum_i e_i(\dot{\vec{r}}_i \cdot \vec{e}_{\vec{k}\sigma}) e^{i\vec{k}\cdot\vec{r}_i}$$

Therefore, we must note that the velocity for a gauge invariant particle is

$$\dot{\vec{r}}_i = \frac{1}{m_i}\big(P_i - e_i\vec{A}(\vec{r}_i)\big)$$

Finally, we write the quantity by using the canonical variables:

$$\begin{aligned}
\vec{E} =& -\dot{\vec{A}} - \vec{\nabla}\phi \\
=& -\frac{1}{\sqrt{\epsilon_0 V}} \sum_{\vec{k}\sigma} \vec{e}_{\vec{k}\sigma} p_{\vec{k}\sigma} e^{-i\vec{k}\cdot\vec{r}} - \vec{\nabla}\phi \\
\vec{B} =& \mathrm{rot}\,\vec{A} \\
=& \frac{i}{\sqrt{\epsilon_0 V}} \sum_{\vec{k}\sigma} \vec{k} \times \vec{e}_{\vec{k}\sigma} q_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}} b
\end{aligned}$$

11.4 Field Momentum

The momentum of the electromagnetic field $\vec{G}_{em}$ is calculated with the Poynting vectors as described in the followings:

[223]For the radiation field:

$$\begin{aligned}
-\dot{p}_{\vec{k}\sigma} &= \frac{\partial H}{\partial q_{\vec{k}\sigma}} \\
&= c^2k^2 q_{-\vec{k}\sigma} + \sum_i \frac{1}{m_i}\left(\vec{P}_i - e_i \frac{1}{\sqrt{\epsilon_0 V}} \sum_{\vec{k}\sigma} \vec{e}_{\vec{k}\sigma} q_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}_i}\right) \cdot \big(-e_i \frac{1}{\sqrt{\epsilon_0 V}} \vec{e}_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}_i}\big) \\
&= c^2k^2 q_{-\vec{k}\sigma} + \sum_i \frac{1}{m_i}(\vec{P}_i - e_i\vec{A}_i) \cdot \big(-e_i \frac{1}{\sqrt{\epsilon_0 V}} \vec{e}_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}_i}\big) \\
&= c^2k^2 q_{-\vec{k}\sigma} - \frac{1}{\sqrt{\epsilon_0 V}} \sum_i e_i(\dot{\vec{r}}_i \cdot \vec{e}_{\vec{k}\sigma}) e^{i\vec{k}\cdot\vec{r}_i} \\
\dot{q}_{\vec{k}\sigma} &= \frac{\partial H}{\partial p_{\vec{k}\sigma}} = p_{-\vec{k}\sigma} \\
\ddot{q}_{-\vec{k}\sigma} &= \dot{p}_{-\vec{k}\sigma} \\
&= -c^2k^2 q_{-\vec{k}\sigma} + \frac{1}{\sqrt{\epsilon_0 V}} \sum_i e_i(\dot{\vec{r}}_i \cdot \vec{e}_{\vec{k}\sigma}) e^{i\vec{k}\cdot\vec{r}_i}
\end{aligned}$$

$$\begin{aligned}
\vec{G} &= \frac{1}{c^2}\int dV\vec{P} = \frac{1}{c^2}\int dV\vec{E}\times\vec{H} \\
&= -\frac{1}{c^2}\frac{1}{\mu_0}\int dV\big(\dot{\vec{A}}+\vec{\nabla}\phi\big)\times \mathrm{rot}\,\vec{A} \\
&= \vec{G}^0_{em}+\vec{G}'_{em} \\
\vec{G}^0_{em} &= -\frac{1}{c^2}\frac{1}{\mu_0}\int dV\,\dot{\vec{A}}\times \mathrm{rot}\,\vec{A} \\
\vec{G}'_{em} &= -\frac{1}{c^2}\frac{1}{\mu_0}\int dV\,\vec{\nabla}\phi\times \mathrm{rot}\,\vec{A}
\end{aligned}$$

The momentum of the pure radiation field $\vec{G}^0_{em}$ can be described using canonical variables: [224]

$$\vec{G}^0_{em} = -i\sum_{\vec{k}\sigma}\vec{k}\,p_{\vec{k}\sigma}q_{\vec{k}\sigma}$$

We further rewrite the terms that are given due to the existing particles. (Note that the boundary terms are cancelled due to the integration by parts and the periodic boundary condition. Note also the Coulomb gauge conditions.):

$$\begin{aligned}
\vec{G}'_{em} &= -\frac{1}{c^2}\frac{1}{\mu_0}\int dV\,\vec{\nabla}\times(\phi\,\mathrm{rot}\,\vec{A})-\phi\,\mathrm{rot}\,\mathrm{rot}\,\vec{A} \\
&= -\frac{1}{c^2}\frac{1}{\mu_0}\int dV\,\phi\,\Delta\vec{A} = -\frac{1}{c^2}\frac{1}{\mu_0}\int dV\,(\Delta\phi)\,\vec{A} \\
&= \frac{1}{c^2}\frac{1}{\epsilon_0\mu_0}\int dV\,\rho\vec{A} = \sum_j e_j\vec{A}_j
\end{aligned}$$

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$$\begin{aligned}
\vec{G}^0_{em} &= -\frac{1}{c^2}\frac{1}{\mu_0}\int dV\,\dot{\vec{A}}\times\mathrm{rot}\,\vec{A} \\
&= -\frac{1}{c^2}\frac{1}{\mu_0}\frac{1}{\sqrt{\epsilon_0}}\sum_{\vec{k}}\sum_{\sigma}\frac{1}{\sqrt{\epsilon_0}}\vec{e}_{\vec{k}\sigma}p_{\vec{k}\sigma}\times(i\vec{k}\times\sum_{\sigma'}\vec{e}_{\vec{k}\sigma'}q_{\vec{k}\sigma'}) \\
&= -i\sum_{\vec{k}}\sum_{\sigma\sigma'}p_{\vec{k}\sigma}q_{\vec{k}\sigma'}\,\vec{e}_{\vec{k}\sigma}\times(\vec{k}\times\vec{e}_{\vec{k}\sigma'}) \\
&= -i\sum_{\vec{k}\sigma}\vec{k}\,p_{\vec{k}\sigma}q_{\vec{k}\sigma}
\end{aligned}$$

$$\vec{e}\times(\vec{k}\times\vec{e}) = \vec{k},\quad (|\vec{e}|=1)$$

Therefore, the total momentum $\vec{G}_T$ is given by the sum of the momentum of the particle system and the momentum of the radiation field:

$$\begin{aligned}\vec{G}_T &= \sum_j m_j \dot{\vec{r}}_j + \vec{G}_{em} \\ &= \sum_j \vec{P}_j + \vec{G}^0_{em}\end{aligned}$$

11.5 Angular Momentum of the Field

Let us calculate for the angular momentum $\vec{J}_{em}$ of the electromagnetic field:

$$\begin{aligned}\vec{J}_{em} &= \frac{1}{c^2}\int dV \vec{r}\times\vec{P} = \frac{1}{c^2}\int dV \vec{r}\times(\vec{E}\times\vec{H}) \\ &= -\frac{1}{c^2}\frac{1}{\mu_0}\int dV \vec{r}\times\left(\dot{\vec{A}}+\vec{\nabla}\phi\right)\times \mathrm{rot}\,\vec{A} \\ &= \vec{J}^0_{em} + J'_{em} \\ \vec{J}^0_{em} &= -\frac{1}{c^2}\frac{1}{\mu_0}\int dV\, \vec{r}\times(\dot{\vec{A}}\times \mathrm{rot}\,\vec{A}) \\ \vec{J}'_{em} &= -\frac{1}{c^2}\frac{1}{\mu_0}\int dV\, \vec{r}\times(\vec{\nabla}\phi\times \mathrm{rot}\,\vec{A})\end{aligned}$$

We divide the angular momentum $\vec{J}^0_{em}$ of the pure radiation field into the following two parts: [225]

[225]

$$\begin{aligned}(\dot{A}\times \mathrm{rot}\,\vec{A})_i =& \epsilon_{ijk}\dot{A}_j\epsilon_{klm}\partial_l A_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\dot{A}_j\partial_l A_m \\ =& \dot{A}_j\partial_i A_j - \dot{A}_j\partial_j A_i = \dot{A}_j\partial_i A_j - \partial_j(\dot{A}_j A_i) + \frac{\partial}{\partial t}(\partial_j A_j)A_i \\ =& \dot{A}_j\partial_i A_j - \partial_j(\dot{A}_j A_i) \\ (\vec{r}\times(\dot{A}\times \mathrm{rot}\,\vec{A}))_a =& \epsilon_{abc}r_b\dot{A}_j\partial_c A_j - \epsilon_{abc}r_b\partial_j(\dot{A}_j A_i) \\ =& \epsilon_{abc}r_b\dot{A}_j\partial_c A_j - \partial_j(\epsilon_{abc}r_b\dot{A}_j A_i) + \epsilon_{abc}\partial_j(r_b)\partial_j(\dot{A}_j A_c) \\ =& \epsilon_{abc}r_b\dot{A}_j\partial_c A_j - \partial_j(\epsilon_{abc}r_b\dot{A}_j A_c) + \epsilon_{abc}\dot{A}_b A_c \\ =& \dot{A}_j(\vec{\ell}A_j)_a - \partial_j(\epsilon_{abc}r_b\dot{A}_j A_c) + \epsilon_{abc}\dot{A}_b A_c\end{aligned}$$

Leave out the boundary terms to obtain

$$\int_V d^3r\,\vec{r}\times(\dot{A}\times \mathrm{rot}\,\vec{A}) = \int_V d^3r\,\dot{A}_j\vec{\ell}A_j + \int_V d^3r\,\dot{\vec{A}}\times\vec{A}$$

$$
\begin{aligned}
\vec{J}_{em} &= \vec{J}^{\ell}_{em} + J^{s}_{em} \\
\vec{J}^{\ell}_{em} &= -\frac{1}{\mu_0 c^2}\int_V d^3r\, \dot{A}_j \vec{\ell} A_j \\
\vec{J}^{s}_{em} &= -\frac{1}{\mu_0 c^2}\int_V d^3r\, \dot{\vec{A}} \times \vec{A} \\
&= -\sum_{k,\sigma\sigma'} (\vec{e}_{k\sigma} \times \vec{e}_{k\sigma'}) p_{k\sigma} q_{k\sigma'}
\end{aligned}
$$

If we conduct the integrations by parts to the terms which arisen from the existence of the particles for a number of times then, we can rewrite the angular momentum into [226]

$$
\vec{J}'_{em} = -\epsilon_0 \int dV \Delta\phi\, \vec{r} \times \vec{A} = \int dV\, \rho \vec{r} \times \vec{A} = \sum_j \vec{r}_j \times (e_j \vec{A}_j)
$$

The angular momentum $\vec{J}_T$ is therefore given by the sum of the angular momentum of the particle system and that of radiation field:

$$
\begin{aligned}
\vec{J}_T &= \sum_j \vec{r}_j \times (m_j \dot{\vec{r}})_j + \vec{J}_{em} = \sum_j \vec{L}_j + \vec{J}^0_{em} \\
\vec{L}_j &= \vec{r}_j \times (m\dot{\vec{r}}_j + e_j A_j) = \vec{r}_j \times \vec{P}_j
\end{aligned}
$$

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$$
\vec{\nabla}\phi \times \mathrm{rot}\,\vec{A} = \vec{\nabla} \times (\phi \mathrm{rot}\,\vec{A}) - \phi \mathrm{rot}\,\mathrm{rot}\,\vec{A} = \vec{\nabla} \times (\phi \mathrm{rot}\,\vec{A}) + \phi \Delta \vec{A}
$$

$$
\begin{aligned}
\vec{r} \times (\vec{\nabla}\phi \times \mathrm{rot}\,\vec{A}) &= \vec{r} \times (\vec{\nabla} \times (\phi \mathrm{rot}\,\vec{A})) + \vec{r} \times \phi \Delta \vec{A} \\
[\vec{r} \times (\vec{\nabla} \times (\phi \mathrm{rot}\,\vec{A}))]_i &= \epsilon_{ijk} r_j \epsilon_{klm} \partial_l (\phi \mathrm{rot}\,\vec{A})_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) r_j \partial_l (\phi \mathrm{rot}\,\vec{A})_m \\
&= r_j \partial_i (\phi \mathrm{rot}\,\vec{A})_j - r_j \partial_j (\phi \mathrm{rot}\,\vec{A})_i \\
&= \partial_i (r_j \phi (\mathrm{rot}\,\vec{A})_j) - \phi (\mathrm{rot}\,\vec{A})_i - \partial_j (r_j \phi (\mathrm{rot}\,\vec{A})_i) + 3\phi (\mathrm{rot}\,\vec{A})_i \\
&= \partial_i (r_j \phi (\mathrm{rot}\,\vec{A})_j) - \partial_j (r_j \phi (\mathrm{rot}\,\vec{A})_i) + 2\phi (\mathrm{rot}\,\vec{A})_i
\end{aligned}
$$

$$
\begin{aligned}
[\vec{r} \times \phi \Delta \vec{A}]_i &= \epsilon_{ijk} r_j \phi \partial_l \partial_l A_k \\
&= \partial_l (\epsilon_{ijk} r_j \phi \partial_l A_k) - \epsilon_{ijk} \phi \partial_j A_k - \epsilon_{ijk} r_j (\partial_l \phi) \partial_l A_k \\
&= \partial_l (\epsilon_{ijk} r_j \phi \partial_l A_k) - \epsilon_{ijk} \phi \partial_j A_k - \partial_l \big(\epsilon_{ijk} r_j (\partial_l \phi) A_k\big) + \epsilon_{ijk} (\partial_j \phi) A_k + \epsilon_{ijk} r_j (\partial_l \partial_l \phi) A_k \\
&= \partial_l (\epsilon_{ijk} r_j \phi \partial_l A_k) - \epsilon_{ijk} \phi \partial_j A_k - \partial_l \big(\epsilon_{ijk} r_j (\partial_l \phi) A_k\big) + \partial_j (\epsilon_{ijk} \phi A_k) - \epsilon_{ijk} \phi (\partial_j A_k) + \epsilon_{ijk} r_j (\partial_l \partial_l \phi) A_k \\
&= \partial_l (\epsilon_{ijk} r_j \phi \partial_l A_k) - \partial_l \big(\epsilon_{ijk} r_j (\partial_l \phi) A_k\big) + \partial_j (\epsilon_{ijk} \phi A_k) - 2\phi (\mathrm{rot}\,\vec{A})_i + (\Delta\phi)(\vec{r} \times \vec{A})_i
\end{aligned}
$$

$$
\vec{J}'_{em} = -\epsilon_0 \int dV \Delta\phi\, \vec{r} \times \vec{A} = \int dV\, \rho \vec{r} \times \vec{A} = \sum_j \vec{r}_j (e_j \times A_j)
$$

12 The Interacting Particle System and Electromagnetic Field as Field Quantity

12.1 Lagrangian Density and Equation of Motion

According to our discussions from the last section, the Maxwell 's equation is

$$\Box\vec{A} = \vec{\nabla}(\mathrm{div}\,\vec{A} + \frac{1}{c^2}\dot{\phi}) - \mu_0\vec{j}$$
$$\frac{1}{c}\Delta\phi = -\frac{1}{c}\frac{\partial}{\partial t}\mathrm{div}\,\vec{A} - \mu_0 c\rho$$

[227] The Maxwell 's equation can be written in the covariant form to the Lorentz transformation:

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \mu_0 j^\nu$$
$$\partial_\mu f^{\mu\nu} = \mu_0 j^\nu$$

[227]

$$\Box\vec{A} = \vec{\nabla}(\mathrm{div}\,\vec{A} + \frac{1}{c^2}\dot{\phi}) - \mu_0\vec{j}$$
$$\frac{1}{c}\Delta\phi = -\frac{1}{c}\frac{\partial}{\partial t}\mathrm{div}\,\vec{A} - \mu_0 c\rho$$

Note:

$$\mathrm{div}\,\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = \partial_\mu A^\mu$$

the first equation is then rewritten as

$$-\partial_\mu\partial^\mu A^i = -\partial^i\partial_\mu A^\mu - \mu_0 j^i$$

While the second equation is rewritten by

$$\Box\frac{1}{c}\phi + \frac{1}{c^3}\frac{\partial\phi}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\partial_\mu A^\mu - \frac{1}{c^2}\frac{\partial}{\partial t}\phi\right) - \mu_0 c\rho$$
$$-\partial_\mu\partial^\mu A^0 = -\partial^0\partial_\mu A^\mu - \mu_0 j^0$$

By organizing the above, the Maxwell 's equation can be written as

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \mu_0 j^\nu$$
$$\partial_\mu f^{\mu\nu} = \mu_0 j^\nu$$

Recall our earlier discussions:

$$
\begin{aligned}
A_0 &= \frac{1}{c}\phi \\
A_1 &= -A^1 = -A_x \\
A_2 &= -A^2 = -A_y \\
A_3 &= -A^3 = -A_z \\
f^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\
j^0 &= c\rho \\
j^i &= (\vec{j})_i
\end{aligned}
$$

The actions derived from the least-action principle, including the actions caused by the particle systems, can be given by the followings. ($\tau_{(i)}$ denotes the eigentime of the ith particle. ($d\tau_{(i)} = dt\sqrt{1-\frac{v_i^2}{c^2}}$):

$$
\begin{aligned}
S_{em} &= S_0 + S_{rad} + S_{el} = \int d^4x \left(\mathcal{L}_0(x) + \mathcal{L}_{rad}(x) + \mathcal{L}_{el}(x)\right) \\
&\quad (d^4x = dx^0dx^1dx^2dx^3 = cdtd^3r) \\
\mathcal{L}_0(x) &= -\sum_i m_i c \int d\tau_{(i)} \sqrt{g_{\mu\nu}\frac{dx^\mu_{(i)}}{d\tau_{(i)}}\frac{dx^\nu_{(i)}}{d\tau_{(i)}}}\,\delta^4(x - x_{(i)}) \\
S_0 &= -\sum_i m_i c \int d\tau_{(i)} \sqrt{g_{\mu\nu}\frac{dx^\mu_{(i)}}{d\tau_{(i)}}\frac{dx^\nu_{(i)}}{d\tau_{(i)}}} = -\sum_i m_i c \int dt\,\sqrt{g_{\mu\nu}\dot{x}^\mu_{(i)}\dot{x}^\nu_{(i)}} \\
\mathcal{L}_{rad}(x) &= -\frac{1}{4\mu_0 c} f_{\mu\nu}f^{\mu\nu} \\
S_{rad} &= -\frac{1}{4\mu_0}\int dtd^3r\, f_{\mu\nu}f^{\mu\nu} \\
\mathcal{L}_{el}(x) &= -j^\mu(x)A_\mu(x) \\
S_{el} &= \int d^4x\,\mathcal{L}_{el}(x) = -\sum_i \int dt\, e_i A_\mu(x_{(i)})\dot{x}^\mu_{(i)} = \sum_i \int dt\, e_i\big(-\phi(\vec{r}_i,t) + \dot{\vec{r}}_i\cdot\vec{A}(\vec{r}_i,t)\big) \\
j^\mu(x) &= \sum_i ce_i \int d\tau_{(i)}\,\delta^4(x - x_{(i)})x'^\mu_{(i)} = (c\sum_i e_i\delta^3(\vec{r}-\vec{r}_i), e_i\dot{\vec{r}}_i\delta^3(\vec{r}-\vec{r}_i))
\end{aligned}
$$

The equation of motion for the radiation field is:

$$
\begin{aligned}
\frac{\delta\mathcal{L}_{rad}}{\delta A_\mu(x)} &= \frac{1}{4\mu_0}\partial_\nu\frac{\partial}{\partial\partial_\nu A_\mu}(\partial_\kappa A_\rho - \partial_\rho A_\kappa)(\partial^\kappa A^\rho - \partial^\rho A^\kappa) \\
&= \frac{1}{\mu_0}\partial_\nu f^{\nu\mu} \\
\frac{\delta\mathcal{L}_{el}}{\delta A_\mu(x)} &= -j^\mu
\end{aligned}
$$

We have already discussed the equation for the particle system.

12.2 Energy-momentum Tensor and the Conservation Law

If we multiply the Maxwell 's equation (field equation) $\partial_\mu f^{\mu\nu} = \mu_0 j^\nu$ by $f_{\lambda\nu}$ along with a further calculation, we obtain: [228]

$$\begin{aligned}\partial_\mu T^\mu{}_\lambda &= f_{\lambda\nu} j^\nu \\ T^\mu{}_\lambda &= \frac{1}{\mu_0}\left(f^{\kappa\mu} f_{\kappa\lambda} - \frac{1}{4}\delta^\mu{}_\lambda f^{\kappa\nu} f_{\kappa\nu}\right)\end{aligned}$$

$T^\mu{}_\lambda$ is called the energy-momentum tensor of electromagnetic field. Specifically, it is $T^{\mu\nu} = g^{\lambda\nu} T^\mu{}_\lambda$:

$$\begin{aligned}T^{\mu\nu} &= \frac{1}{\mu_0}\left(g^{\lambda\nu} g_{\kappa\alpha} g_{\lambda\beta} f^{\kappa\mu} f^{\alpha\beta} - \frac{1}{4} g^{\lambda\nu}\delta^\mu{}_\lambda f^{\kappa\nu} f_{\kappa\nu}\right) \\ &= \frac{1}{\mu_0}\left(g_{\kappa\alpha} f^{\kappa\mu} f^{\alpha\nu} - \frac{1}{4} g^{\mu\nu} f^{\kappa\nu} f_{\kappa\nu}\right)\end{aligned}$$

What we described in the above is symmetric to $T^{\mu\nu} = T^{\nu\mu}$ and therefore the

[228]

$$\begin{aligned}f_{\lambda\nu}\partial_\mu f^{\mu\nu} &= \partial_\mu(f_{\lambda\nu} f^{\mu\nu}) - f^{\mu\nu}\partial_\mu f_{\lambda\nu} \\ &= \partial_\mu(f_{\lambda\nu} f^{\mu\nu}) - \frac{1}{2} f^{\mu\nu}(\partial_\mu f_{\lambda\nu} - \partial_\nu f_{\lambda\mu}), \quad f^{\mu\nu} = -f^{\nu\mu} \\ &= \partial_\mu(f_{\lambda\nu} f^{\mu\nu}) - \frac{1}{2} f^{\mu\nu}(\partial_\mu f_{\lambda\nu} + \partial_\nu f_{\mu\lambda} + \partial_\lambda f_{\nu\mu}) + \frac{1}{2} f^{\mu\nu}\partial_\lambda f_{\nu\mu} \\ &= \partial_\mu(f_{\lambda\nu} f^{\mu\nu}) + \frac{1}{2} f^{\mu\nu}\partial_\lambda f_{\nu\mu} \\ &= \partial_\mu(f_{\lambda\nu} f^{\mu\nu}) - \frac{1}{4}\partial_\lambda(f^{\mu\nu} f_{\mu\nu}) = \partial_\mu(f_{\lambda\nu} f^{\mu\nu}) - \frac{1}{4}\partial_\lambda(f^{\kappa\nu} f_{\kappa\nu}) \\ &= \partial_\mu(f_{\lambda\nu} f^{\mu\nu}) - \frac{1}{4}\delta^\mu{}_\lambda\partial_\mu(f^{\kappa\nu} f_{\kappa\nu}) \\ &= \partial_\mu\left(f^{\kappa\mu} f_{\kappa\lambda} - \frac{1}{4}\delta^\mu{}_\lambda f^{\kappa\nu} f_{\kappa\nu}\right)\end{aligned}$$

Thus,

$$\partial_\mu f_{\lambda\nu} + \partial_\nu f_{\mu\lambda} + \partial_\lambda f_{\nu\mu} = \partial_\mu(\partial_\lambda A_\nu - \partial_\nu A_\lambda) + \partial_\nu(\partial_\mu A_\lambda - \partial_\lambda A_\mu) + \partial_\lambda(\partial_\nu A_\mu - \partial_\mu A_\nu) = 0$$

energy-momentum tensor is expressed in the form: [229]

$$T^{00} = -\frac{1}{2}(\epsilon_0 \vec{E}^2 + \mu_0 \vec{H}^2) = -\mathcal{H}_{em}$$
$$T^{k0} = -\frac{1}{c}(\vec{P})_k, \quad \vec{P} = \vec{E} \times \vec{H}$$
$$T^{kl} = \epsilon_0 E_k E_l + \mu_0 H_k H_l - \delta_{kl}\frac{1}{2}(\epsilon_0 \vec{E}^2 - \mu_0 \vec{H}^2))$$

Note also:

$$\partial_\mu T^{\mu\kappa} = f^{\kappa\nu} j_\nu$$

If the equation of motion for ith particle is written by $\frac{d\pi^\mu_{(i)}}{dt} = e_i \dot{x_{\kappa(i)}} f^{\mu\kappa}$, then we can write:

$$\int_V d^3r\, j_(x) = \sum_i e_i \dot{x_{\kappa(i)}} f^{\mu\kappa}$$

[229]

$$f_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & 0 & B_x & 0 \end{pmatrix}_{\mu\nu}$$

$$f^{\alpha\beta} = g^{\alpha\mu} g^{nu\beta} f_{\mu\nu}$$
$$= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & 0 & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\}_{\alpha\beta}$$
$$= \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}_{\alpha\beta}$$

which gives

$$f^{\alpha\beta} f_{\alpha\beta} = -\frac{2}{c^2}\vec{E}^2 + 2\vec{B}^2$$

$\mathrm{T}^{k0} = -\frac{1}{c}\vec{P}_k, \quad \vec{P} = \vec{E} \times \vec{H}$
$Further, \quad T^{kl} = \frac{1}{\mu_0}(\frac{1}{c^2} E_k E_l + B_k B_l + \delta_{kl}\frac{1}{2}(-\frac{1}{c^2}\vec{E}^2 + \vec{B}^2))$
$= \epsilon_0 E_k E_l + \mu_0 H_k H_l - \delta_{kl}\frac{1}{2}(\epsilon_0 \vec{E}^2 - \mu_0 \vec{H}^2))$

Having mentioned the above: [230]

$$\frac{d}{dt}\sum_i \pi^{\mu}_{(i)} = \frac{1}{c}\frac{\partial}{\partial t}\int_V d^3r\, T^{0\mu}$$

The expressions in each component:

$$\sum_i M_i c^2 + \int_V d^3r\, \mathcal{H}_{em}(\vec{r}) = const.$$
$$\sum_i M_i \vec{v}_i + \int_V d^3r\, \vec{P}(\vec{r}) = const.$$

which represent the conservation of momentum and energy.

13 Quantization of Electromagnetic Field and the Charged Particles

We conduct quantization of the system as we follow the classic canonical equation we obtained in the previous section. For the operators, we have the canonical variables in the radiation field $q_{\vec{k}\sigma}$ and $p_{\vec{k}\sigma}$, and the canonical variable of the particle system $\vec{r}_i$ and its conjugate momentum $\vec{P}_i = m_i\dot{\vec{r}}_i + e_i\vec{A}$. The commutation relation is imposed between the operators:

$$\begin{aligned} [q_{\vec{k}\sigma}, p_{\vec{k}'\sigma'}] &= i\hbar\delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'} \\ [r_i^{\alpha}, P_j^{\beta}] &= i\hbar\delta_{ij}\delta_{\alpha\beta} \end{aligned}$$

To clarify more, we use a differential representation for the particle system:

$$\vec{P}_i = -i\hbar\vec{\nabla}_i$$

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$$\begin{aligned} \frac{d}{dt}\sum_i \pi^{\mu}_{(i)} &= \int_V d^3r\, \partial_\nu T^{\nu\mu} \\ &= \frac{1}{c}\frac{\partial}{\partial t}\int_V d^3r\, T^{0\mu} + \int_V \partial_i T^{i\mu} \\ &= \frac{1}{c}\frac{\partial}{\partial t}\int_V d^3r\, T^{0\mu} + \int_S dS_i T^{i\mu} = \frac{1}{c}\frac{\partial}{\partial t}\int_V d^3r\, T^{0\mu} \end{aligned}$$

For the radiation field, we express in boson representation:

$$
\begin{aligned}
q_{\vec{k}\sigma} &= \sqrt{\frac{\hbar}{2\omega_k}}(a^\dagger_{-\vec{k}\sigma} + a_{\vec{k}\sigma}) \\
p_{\vec{k}\sigma} &= i\sqrt{\frac{\hbar\omega_k}{2}}(a^\dagger_{\vec{k}\sigma} - a_{-\vec{k}\sigma}) \\
[a_{\vec{k}\sigma}, a^\dagger_{\vec{k}'\sigma'}] &= \delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'} \\
[a_{\vec{k}\sigma}, a_{-\vec{k}'\sigma'}] &= 0 \\
[a^\dagger_{\vec{k}\sigma}, a^\dagger_{-\vec{k}'\sigma'}] &= 0
\end{aligned}
$$

The vector potential can be written by using the representations above: [231]

$$
\vec{A}(\vec{r}) = \frac{1}{\sqrt{\epsilon_o V}} \sum_{\vec{k}\sigma} \sqrt{\frac{\hbar}{2\omega_k}} \vec{e}_{\vec{k}\sigma}(a^\dagger_{\vec{k}\sigma} e^{-i\vec{k}\cdot\vec{r}} + a_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}})
$$

[231]

$$
\begin{aligned}
\vec{A}(\vec{r}) &= \frac{1}{\sqrt{\epsilon_o V}} \sum_{\vec{k}\sigma} \vec{e}_{\vec{k}\sigma} q_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}} \\
&= \frac{1}{\sqrt{\epsilon_o V}} \sum_{\vec{k}\sigma} \sqrt{\frac{\hbar}{2\omega_k}} \vec{e}_{\vec{k}\sigma}(a^\dagger_{-\vec{k}\sigma} + a_{\vec{k}\sigma}) e^{i\vec{k}\cdot\vec{r}} \\
&= \frac{1}{\sqrt{\epsilon_o V}} \sum_{\vec{k}\sigma} \sqrt{\frac{\hbar}{2\omega_k}} \vec{e}_{\vec{k}\sigma}(a^\dagger_{\vec{k}\sigma} e^{-i\vec{k}\cdot\vec{r}} + a_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}})
\end{aligned}
$$

Commutation Relation of a Field Quantity

Here, we calculate for the commutation relations of the field quantity: [232]

$$
\begin{aligned}
[A_\alpha(\vec r), A_\beta(\vec r\,')] =& 0\\
[E_\alpha(\vec r), E_\beta(\vec r\,')] =& 0\\
[B_\alpha(\vec r), B_\beta(\vec r\,')] =& 0\\
[E_\alpha(\vec r), A_\kappa(\vec r\,')] =& i\hbar\frac{1}{\epsilon_0 V}\epsilon_{\alpha\beta\gamma}\partial'_\gamma\delta(\vec r-\vec r\,')
\end{aligned}
$$

13.1 Hamiltonian

The Hamiltonian of the classical system therefore can be rewritten by the operators we have defined: [233]

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$$
\begin{aligned}
[A_\alpha(\vec r), A_\beta(\vec r\,')] =& 0\\
[E_\alpha(\vec r), E_\beta(\vec r\,')] =& 0\\
[B_\alpha(\vec r), B_\beta(\vec r\,')] =& 0\\
[E_\alpha(\vec r), A_\beta(\vec r\,')] =& -\frac{1}{\epsilon_0 V}\sum_{\vec k\sigma}(\vec e_{\vec k\sigma})_\alpha(\vec e_{\vec k\sigma})_\beta[p_{\vec k\sigma}, q_{\vec k\sigma}]e^{-i\vec k\cdot(\vec r-\vec r\,')}\\
=& \frac{i\hbar}{\epsilon_0 V}\sum_{\vec k\sigma}(\vec e_{\vec k\sigma})_\alpha(\vec e_{\vec k\sigma})_\beta e^{i\vec k\cdot(\vec r-\vec r\,')}\\
=& \frac{i\hbar}{\epsilon_0 V}\sum_{\vec k}(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2})e^{i\vec k\cdot(\vec r-\vec r\,')}\\
[E_\alpha(\vec r), B_\beta(\vec r\,')] =& \epsilon_{\beta\gamma\kappa}\partial'_\gamma[E_\alpha(\vec r), A_\kappa(\vec r\,')]\\
=& -\frac{\hbar}{\epsilon_0 V}\sum_{\vec k}(\delta_{\alpha\kappa}-\frac{k_\alpha k_\kappa}{k^2})\epsilon_{\beta\gamma\kappa}k_\gamma e^{i\vec k\cdot(\vec r-\vec r\,')}\\
=& -\frac{\hbar}{\epsilon_0 V}\sum_{\vec k}\epsilon_{\beta\gamma\alpha}k_\gamma e^{i\vec k\cdot(\vec r-\vec r\,')}\\
=& i\frac{\hbar}{\epsilon_0 V}\epsilon_{\alpha\beta\gamma}\partial'_\gamma\delta(\vec r-\vec r\,')
\end{aligned}
$$

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$$
\begin{aligned}
\frac{1}{2}\sum_k\left(p_{\vec k\sigma}p_{-\vec k\sigma}+\omega_k^2 q_{\vec k\sigma}q_{-\vec k\sigma}\right) &= \sum_k\frac{\hbar\omega_k}{4}\Big(-(a^\dagger_{\vec k\sigma}-a_{-\vec k\sigma})(a^\dagger_{-\vec k\sigma}-a_{\vec k\sigma})+(a^\dagger_{-\vec k\sigma}+a_{\vec k\sigma})(a^\dagger_{\vec k\sigma}+a_{-\vec k\sigma})\Big)\\
&= \sum_k\hbar\omega_k\frac{1}{4}(a_{-\vec k\sigma}a^\dagger_{-\vec k\sigma}+a^\dagger_{\vec k\sigma}a_{\vec k\sigma})+a^\dagger_{-\vec k\sigma}a_{-\vec k\sigma})+a_{\vec k\sigma}a^\dagger_{\vec k\sigma})\\
&= \sum_k\hbar\omega_k\frac{1}{2}(a_{\vec k\sigma}a^\dagger_{\vec k\sigma}+a^\dagger_{\vec k\sigma}a_{\vec k\sigma})\\
&= \sum_k\hbar\omega_k(a^\dagger_{\vec k\sigma}a_{\vec k\sigma}+\frac{1}{2})
\end{aligned}
$$

[234]

$$
\begin{aligned}
H &= H_{part} + H_{rad} + H_{coulomb} \\
H_{part} &= \sum_i \frac{1}{2m_i}(-i\hbar\vec{\nabla}_i - e_i\vec{A}(\vec{r}_i))^2 \\
\vec{A}(\vec{r}_i) &= \frac{1}{\sqrt{\epsilon_o V}} \sum_{\vec{k}\sigma} \sqrt{\frac{\hbar}{2\omega_k}} \vec{e}_{\vec{k}\sigma}(a^\dagger_{\vec{k}\sigma} e^{-i\vec{k}\cdot\vec{r}_i} + a_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}_i}) \\
H_{rad} &= \sum_{\vec{k}} \sum_{\sigma=1,2} \hbar\omega_k (n_{\vec{k}\sigma} + \frac{1}{2}) \\
n_{\vec{k}\sigma} &= a^\dagger_{\vec{k}\sigma} a_{\vec{k}\sigma'} \\
H_{coulomb} &= \sum_i \frac{e_i e_j}{|\vec{r}_i - \vec{r}_j|}
\end{aligned}
$$

13.2 Momentum

Here, the field momentum is written as [235]

$$
\vec{G}^0_{em} = \sum_{\vec{k}\sigma} \hbar\vec{k}\; n_{\vec{k}\sigma}
$$

---

[234]

$$
\vec{A}(\vec{r}_i) = \frac{1}{\sqrt{\epsilon_o V}} \sum_{\vec{k}\sigma} \sqrt{\frac{\hbar}{2\omega_k}} \vec{e}_{\vec{k}\sigma}(a^\dagger_{\vec{k}\sigma} e^{-i\vec{k}\cdot\vec{r}_i} + a_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}_i})
$$

[235]

$$
\begin{aligned}
G^0_{em} &= -i \sum_{\vec{k}\sigma} \vec{k}\; p_{\vec{k}\sigma} q_{\vec{k}\sigma} \\
&= \frac{1}{2} \sum_{\vec{k}\sigma} \hbar\vec{k}(a^\dagger_{\vec{k}\sigma} - a_{-\vec{k}\sigma})(a^\dagger_{-\vec{k}\sigma} + a_{\vec{k}\sigma}) \\
&= \frac{1}{2} \sum_{\vec{k}\sigma} \hbar\vec{k}(a^\dagger_{\vec{k}\sigma} a^\dagger_{-\vec{k}\sigma} - a_{-\vec{k}\sigma} a_{\vec{k}\sigma} + a^\dagger_{\vec{k}\sigma} a_{\vec{k}\sigma} - a_{-\vec{k}\sigma} a^\dagger_{-\vec{k}\sigma}) \\
&= \sum_{\vec{k}\sigma} \hbar\vec{k}\; a^\dagger_{\vec{k}\sigma} a_{\vec{k}\sigma} \quad (\vec{k} \leftrightarrow -\vec{k})
\end{aligned}
$$

Note that we have $(\vec{k} \leftrightarrow -\vec{k})$ in the last form above.

The momentum of the particle is added to the equation above, and we further write

$$\begin{aligned}\vec{G}_T &= \vec{G}_p + \vec{G}^0_{em} \\ &= \sum_i \frac{\hbar}{i}\vec{\nabla}_i + \sum_{\vec{k}\sigma} \hbar\vec{k}\, n_{\vec{k}\sigma} \\ \vec{G}_p &= \sum_i \frac{\hbar}{i}\vec{\nabla}_i\end{aligned}$$

We can also show that the commutators of the momentum and the Hamiltonian to be defined as [236]

$$[H, \vec{G}_T] = 0$$

14 Interaction of Electromagnetic Field with Matter

We now consider the terms A and A^2 as the perturbation Hamiltonian since the absence of the terms causes the particle system and the radiation field to be separated from one another. In our following discussions, we apply the perturbation theory in considering the issue here. The two terms are in the Coulomb gauge: [237]

$$\vec{P}_i \cdot \vec{A}(\vec{r}_i) = \vec{A}(\vec{r}_i) \cdot \vec{P}_i$$

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$$\begin{aligned}[e^{i\vec{k}\cdot\vec{r}_j}, \vec{\nabla}_j] &= -i\vec{k}e^{i\vec{k}\cdot\vec{r}_j} \\ [a, a^\dagger a] &= a \\ [a^\dagger, a^\dagger a] &= -a^\dagger \\ [(\vec{A}(\vec{r}_i))_\alpha, \vec{G}_T] &= \frac{1}{\sqrt{\epsilon_0 V}} \sum_{\vec{k}\sigma} \sqrt{\frac{\hbar}{2\omega_k}} (\vec{e}_{\vec{k}\sigma})_\alpha \Big([a^\dagger_{\vec{k}\sigma} e^{-i\vec{k}\cdot\vec{r}_i}, \frac{\hbar}{i}\vec{\nabla}_i + \hbar\vec{k}n_{\vec{k}\sigma}] \\ &\quad + [a_{\vec{k}\sigma} e^{i\vec{k}\cdot\vec{r}_i}, \frac{\hbar}{i}\vec{\nabla}_i + \hbar\vec{k}n_{\vec{k}\sigma}] \Big) = 0 \\ [H_{part}, \vec{G}_T] &= 0 \\ [H, \vec{G}_T] &= [H_{part} + H_{rad} + H_{coulomb}, \vec{G}_p + \vec{G}^0_{em}] \\ &= [H_{rad} + H_{coulomb}, \vec{G}_p + \vec{G}^0_{em}] \\ &= [H_{coulomb}, \vec{G}_p + \vec{G}^0_{em}] \\ &= [H_{coulomb}, \vec{G}_p] = 0\end{aligned}$$

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$$[\vec{P}_i, \vec{A}(\vec{r}_i)]* = \vec{A}_i \cdot \vec{P}_i(*) + (\vec{P}_i \cdot \vec{A}_i) * - \vec{A}_i(\cdot \vec{P}_i *) = -i\hbar \mathrm{div}\, \vec{A}(\vec{r}_i) = 0$$

Having aware of the fact above, separate the Hamiltonian in the system:

$$H = H_0 + H_{int}$$

Here H_0 is the Hamiltonian described in the following with the particle system and the radiation field being separated:

$$\begin{aligned} H_0 &= H_p + H_{rad} \\ H_p &= -\sum_i \frac{\hbar^2}{2m_i}\Delta_i + \sum_i \frac{e_i e_j}{|\vec{r}_i - \vec{r}_j|} \\ H_{rad} &= \sum_{\vec{k}} \sum_{\sigma=1,2} \hbar\omega_k (n_{\vec{k}\sigma} + \frac{1}{2}) \end{aligned}$$

While H_{int} denotes the interaction between particle system and the radiation field due to the vector potential:

$$\begin{aligned} H_{int} &= H^{(1)} + H^{(2)} \\ H^{(1)} &= \sum_i \frac{i\hbar e_i}{m_i} \vec{A}(\vec{r}_i) \cdot \vec{\nabla}_i \\ &= \frac{1}{\sqrt{\epsilon_o V}} \sum_i \frac{i\hbar e_i}{m_i} \sum_{\vec{k}\sigma} \sqrt{\frac{\hbar}{2\omega_k}} (a^\dagger_{-\vec{k}\sigma} + a_{\vec{k}\sigma}) e^{i\vec{k}\cdot\vec{r}_i} \, (\vec{e}_{\vec{k}\sigma} \cdot \vec{\nabla}_i) \\ H^{(2)} &= \sum_i \frac{\hbar (e_i)^2}{2m_i} \vec{A}(\vec{r}_i)^2 \\ &= \sum_i \frac{\hbar (e_i)^2}{2m_i} \frac{1}{\epsilon_o V} \sum_{\vec{k}\vec{k}'\sigma\sigma'} \frac{\hbar(\vec{e}_{\vec{k}\sigma} \cdot \vec{e}_{\vec{k}'\sigma'})}{2\sqrt{\omega_k \omega_{k'}}} (a^\dagger_{-\vec{k}\sigma} + a_{\vec{k}\sigma})(a^\dagger_{-\vec{k}'\sigma'} + a_{\vec{k}'\sigma'}) e^{i(\vec{k}\cdot\vec{r}_i + \vec{k}'\cdot\vec{r}_i')} \end{aligned}$$

The non-perturbation basis set can be written by the eigenstate $\Psi_m(\{\vec{r}_i\})$ and eigenenergy E_m of the particle system, as well as by the state vector $|\{n_{\vec{k}\sigma}\}\rangle$ of the radiation field. (We exclude the zero-point energy in this case.):

$$\begin{aligned} H_0|m; \{n_{\vec{k}\sigma}\}\rangle &= (E_m + \sum_{\vec{k}\sigma} n_{\vec{k}} \hbar\omega_{\vec{k}}))|m; \{n_{\vec{k}\sigma}\}\rangle \\ |m; \{n_{\vec{k}\sigma}\}\rangle &= |\{n_{\vec{k}\sigma}\}\rangle \Psi_m(\{\vec{r}_i\}) \\ H_p \Psi_m(\{\vec{r}_i\} &= E_m \Psi_m(\{\vec{r}_i\}) \\ H_{rad}|\{n_{\vec{k}\sigma}\}\rangle &= \sum_{\vec{k}\sigma} n_{\vec{k}} \hbar\omega_{\vec{k}} |\{n_{\vec{k}\sigma}\}\rangle \end{aligned}$$

Note that $H^{(1)}$ denotes the photon absorption and emission while $H^{(2)}$ denotes the process involving the two photons. So far, we have ignored the relativity effects on the particle system but since we recognize the lowest order relativity correction

$$-\frac{e\hbar}{2m} \vec{\sigma} \cdot \mathrm{rot}\, \vec{A}$$

the following term then be added to the perturbation Hamiltonian:

$$
\begin{aligned}
H^{(s)} &= -\sum_i \frac{e_i\hbar}{2m_i}\vec{\sigma}\cdot \mathrm{rot}_i\,\vec{A}_i = -\sum_i \frac{e_i\hbar}{2m_i}\vec{\sigma}\cdot\vec{\nabla}_i\times\vec{A}_i \\
&= -\frac{1}{\sqrt{\epsilon_0 V}}\sum_i\sum_{k,\sigma}\frac{ie_i\hbar}{2m_i}\sqrt{\frac{\hbar}{2\omega_k}}(a^\dagger_{-\vec{k}\sigma}+a_{\vec{k}\sigma})e^{i\vec{k}\cdot\vec{r}_i}\vec{\sigma}\cdot(\vec{e}_{\vec{k}\sigma}\times\vec{k})
\end{aligned}
$$

14.1 Fermi 's Golden Rule

Let us review the Fermi 's golden rule that relates to the transition probability of the states calculated by the perturbation theory. We consider the non-perturbation system and its state:

$$H_0|n\rangle = E_n|n\rangle$$

Then, we suppose all system to be governed by (time independent) Hamiltonian:

$$H = H_0 + H_{int}$$

We determine the probability of transition per unit time from the time zero of non-perturbation state a to the non-perturbation state b. We assume in this case that the perturbation is small enough while having sufficient observation time.

- Interaction representation

  Schroedinger equation:

$$i\hbar\partial_t\Psi = (H_0 + H_{int})\Psi$$

  From the equation above, we let [238]

$$\Psi = e^{-iH_0t/\hbar}\Psi^{\mathrm{I}}$$

  giving

$$
\begin{aligned}
i\hbar\partial_t\Psi^{\mathrm{I}} &= H^{\mathrm{I}}_{int}\Psi^{\mathrm{I}} \\
H^{\mathrm{I}}_{int} &= e^{iH_0t/\hbar}H_{int}e^{-iH_0t/\hbar}
\end{aligned}
$$

  This is known as the interaction representation. Now, we write

$$\Psi^{\mathrm{I}}(t) = \sum_n c_n(t)|n\rangle$$

[238] Make substitution.

and which gives:

$$\begin{aligned} i\hbar\dot{c}_n &= \sum_m \langle n|H^{\mathrm{I}}_{int}|m\rangle c_m \\ &= \sum_m \langle n|H_{int}|m\rangle e^{i(E_n-E_m)t/\hbar} c_m \end{aligned}$$

Thus, we can simply derive the conservation of probability (self-evident?):

$$\frac{d}{dt}\sum_n |c_n(t)|^2 = 0$$

Now, go back to our initial discussion where

$$c_a(t=0) = 1, \quad c_n(t=0) = 0, \ (n \neq a)$$

and we suppose only a very little time has elapsed from the initial condition. The successive approximate solution can be obtained by [239]

$$c_b(t) = \langle b|H_{int}|a\rangle \frac{e^{i(E_b-E_a)t/\hbar} - 1}{E_b - E_a}$$

which gives

$$|c_b(t)|^2 = |\langle b|H_{int}|a\rangle|^2 2\frac{\cos(E_b-E_a)t/\hbar}{(E_b-E_a)^2}$$

Here, if we use [240] [241]

$$\delta(x) = \lim_{\alpha\to\infty} \frac{1-\cos\alpha x}{\pi\alpha x^2}$$

the probability of transition $w_{a\to b}$ from a to b per unit time can be given by the following: [242]

$$w_{a\to b} = \frac{1}{t}|c_b(t)|^2 \longrightarrow \frac{2\pi}{\hbar}|\langle b|H_{int}|a\rangle|^2 \delta(E_b - E_a)$$

---

[239]

$$i\hbar\dot{c}_b(t) = \langle b|H_{int}|a\rangle e^{i(E_b-E_a)t/\hbar} c_a$$

[240] The effective range of the successive approximation will be

$$|\langle b|H_{int}|a\rangle| << |E_b - E_a|$$

We also know that this is time independent.

[241]

$$\int_{-\infty}^{\infty} dy \frac{1-\cos\alpha x}{y^2} = \pi$$

[242] The validity of the substitution in the delta function can be proven by

$$\frac{|E_a - E_b|t}{\hbar} >> 1$$

In other words, the transition occurs between the different states with the same energy. If the final state b, for example, belongs to the continuous spectrum with the density of states $\rho(E_b)$ at energy interval dE_b, there will be $\rho(E_b)dE_b$ states, thereby the transition probability can be

$$\int w_{a\to b}\rho(E_b)dE_b = \frac{2\pi}{\hbar}|\langle b|H_{int}|a\rangle|^2\rho(E_b)$$

This is known as the Fermi 's golden rule. [243]

14.2 Transition Matrix Elements and Dipole Transition

We now discuss the absorption and emission of light exclusively to during the first order where we can apply the Fermi 's golden rule. To do so, we must calculate the following matrix elements: [244]

$$\begin{aligned}
\langle m_b; \{n_{\vec{k}\sigma}\}_b|H^{(1)}||m_a; \{n_{\vec{k}\sigma}\}_a\rangle &= \sum_{\vec{k}\sigma} M^p_{ba}(\vec{k},\sigma)M^{rad}_{ba}(\vec{k},\sigma) \\
M^p_{ba}(\vec{k},\sigma) &= = \int\prod d\vec{r}_i\Psi^*_b(\{\vec{r}_i\})\left(\sum_i \frac{i\hbar e_i}{m_i}e^{i\vec{k}\cdot\vec{r}_i}(\vec{e}_{\vec{k}\sigma}\cdot\vec{\nabla}_i)\right)\Psi_a(\{\vec{r}_i\}) \\
M^{rad}_{ba}(\vec{k},\sigma) &= \frac{1}{\sqrt{\epsilon_o V}}\sqrt{\frac{\hbar}{2\omega_k}}\langle\{n_{\vec{k}\sigma}\}_b|(a^\dagger_{-\vec{k}\sigma}+a_{\vec{k}\sigma})|\{n_{\vec{k}\sigma}\}_a\rangle
\end{aligned}$$

We use the following evaluation for the radiation field:

$$\begin{aligned}
\sqrt{\frac{\hbar}{2\omega}}\langle n-1|a|n\rangle &= \sqrt{\frac{\hbar}{2\omega}}\sqrt{n} \\
\sqrt{\frac{\hbar}{2\omega}}\langle n+1|a^\dagger|n\rangle &= \sqrt{\frac{\hbar}{2\omega}}\sqrt{n+1}
\end{aligned}$$

Now consider the matrix element $M^p_{ba}(\vec{k}\sigma)$ given by the wavefunction $\Psi_m(\{\vec{r}_i\})$ of the particle system $(m = a, b)$. We let the radius of an atom be a to estimate the energy difference E for before and after the transition, thereby supposing the bound energy of the atom as

$$E = \hbar\omega \approx \frac{e^2}{4\pi\epsilon_0 a}$$

which gives the wave number k of the related light:

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \frac{E}{\hbar c} \approx \frac{1}{a}\frac{e^2}{4\pi\epsilon_0\hbar c} = \alpha\frac{1}{a}$$

[243] The approximation.
[244] Consider the fermion system. For the boson system, normalization must be considered.

Thus,

$$k \approx \frac{\alpha}{a} << \frac{1}{a}, \quad \alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137},$$

Note that α is the non-dimensional physical constant called the fine-structure constant. In the region where the wavefunction of the particle system possesses the finite values, we can only consider the wave number of the light and

$$\vec{k} = 0$$

Further, we write the following for the Hamiltonian H_p of the particle system: [245]

$$[H_p, \vec{r}_i] = -\frac{\hbar^2}{m}\vec{\nabla}_i$$
$$[H_p, r_{i,\alpha}] = -\frac{\hbar^2}{m}\partial_{i,\alpha}$$

Since the states are the eigenstates of the Hamiltonian:

$$\begin{aligned}
M^p_{ba} &\approx M^{p,e-dipole}_{ba} \\
M^{p,e-dipole}_{ba} &= (E_b - E_a)\int \prod d\vec{r}_i \Psi^*_b(\{\vec{r}_i\})\left(\sum_i (\vec{e}_{\vec{k}=0,\sigma}\cdot\vec{r}_i)\frac{ie_i}{\hbar}\right)\Psi_a(\{\vec{r}_i\}) \\
&= (E_b - E_a)\langle b|\sum_i (\vec{e}_{\vec{k}=0,\sigma}\cdot\vec{r}_i)\frac{ie_i}{\hbar}|a\rangle \\
&= -i\omega_{ba}\mu^T_{\sigma,ba} \\
\mu^T_{\sigma,ba} &= \sum_i \langle b|\mu^i_\sigma|a\rangle, \qquad \hbar\omega_{ba} = E_b - E_a \\
&\langle b|\cdots|a\rangle \equiv \int \prod d\vec{r}_i \Psi^*_b(\{\vec{r}_i\})(\cdots)\Psi_a(\{\vec{r}_i\}), \\
\mu^i_\sigma &= \vec{e}_{\vec{k},\sigma}\cdot\vec{\mu}_i, \quad \vec{\mu}_i = e_i\vec{r}_i \ \ (electricdipole)
\end{aligned}$$

The approximation at $e^{i\vec{k}\cdot r_i} \to 1$ is called the electric dipole. Commonly, the oscillator strength f_{ba} is defined so as to express the magnitude of the transition for $b \to a$:

$$f_{ab} = \frac{2m}{e^2\hbar\omega_{ba}}|M^p_{ba}|^2$$

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$$[\frac{p^2}{2m}, r] = \frac{p}{2m}2[p,r] = \frac{p}{2m}2(-)i\hbar = -ip\frac{\hbar}{m}$$

The total sum rule is satisfied by the oscillator strength of the electric dipole transition: [246]

$$\sum_b f_{ba} = N$$

Some transitions particularly provide essential contributions, and those contributions are considered as $\mathcal{O}(1)$.

If there is a zero contribution from the electric dipole approximation due to the symmetry, the degree of order described in the following must be considered. Here, assume $e^{i\vec{k}\cdot r_i} \to 1 + i\vec{k}\cdot\vec{r}_i$,

$$M^p_{ba} \approx M^{p,e-d}_{ba} + \int \prod d\vec{r}_i \Psi^*_b(\{\vec{r}_i\}) \left(\frac{i\hbar e_i}{m_i} \sum_i i\vec{k}\cdot\vec{r}_i(\vec{e}_{\vec{k},\sigma}\cdot\vec{\nabla}_i) \right) \Psi_a(\{\vec{r}_i\})$$

[246] First, confirm the double commutator in the following:

$$\begin{aligned}
[\sum_i^N r_{i,\alpha}, [H_p, \sum_j^N r_{j,\beta}]] =& \frac{1}{2m}[\sum_i^N r_{i,\alpha}, [\sum_k^N \vec{p}_k^2, \sum_j^N r_{j,\beta}]] \\
=& -2i\hbar\frac{1}{2m}[\sum_i^N r_{i,\alpha}, \sum_j^N p_{j,\beta}] \\
=&(-2i\hbar)(i\hbar)\frac{1}{2m}N\delta_{\alpha\beta} = \frac{\hbar^2}{m}N\delta_{\alpha\beta} \\
[\sum_i^N(\vec{e}_\sigma\cdot\vec{r}_i), [H_p, \sum_j^N(\vec{e}_\sigma\cdot\vec{r}_j)]] =&(\vec{e}_\sigma)_\alpha(\vec{e}_\sigma)_\alpha\frac{\hbar^2}{m}N = \frac{\hbar^2}{m}N
\end{aligned}$$

$[x,[H,x]] = [x, Hx - xH] = xHx - x^2H - Hx^2 + xHx = 2xHx - x^2H - Hx^2$ gives

$$\begin{aligned}
\langle a|[x,[H,x]]|a\rangle =& 2\langle a|xHx|a\rangle - \langle a|x^2H|a\rangle - \langle a|Hx^2|a\rangle \\
=& 2\langle a|xHx|a\rangle - E_a\langle a|x^2|a\rangle - E_a\langle a|x^2|a\rangle \\
=& 2\sum_b \langle a|x|b\rangle\langle b|Hx|a\rangle - 2E_a\sum_b\langle a|x|b\rangle\langle b|x|a\rangle \\
=& 2\sum_b (E_b - E_a)|\langle b|x|a\rangle|^2
\end{aligned}$$

Thus, given $x = \sum_i \vec{e}_\sigma\cdot\vec{r}_i$, we use $\langle a|a\rangle = 1$ and the completeness of the intermediate state:

$$\sum_b f_{ba} = \sum_b \frac{2m}{e^2\hbar}\omega_{ba}|\mu^T_{\sigma,ba}|^2 = \sum_b \frac{2}{e^2m\hbar^2}(E_b - E_a)|\mu^T_{\sigma,ba}|^2 = N$$

Now, [247]

$$(\vec{k}\cdot\vec{r})(\vec{e}\cdot\vec{\nabla}) = \frac{1}{2}(\vec{k}\times\vec{e})\cdot\vec{\ell} + \frac{1}{2}[H_p, (\vec{k}\cdot\vec{r})(\vec{e}\cdot\vec{r})]$$

which provides

$$M^{p}_{ba} \approx M^{p,e-d}_{ba} + M^{p,e-q}_{ba} + M^{p,m-d_2}_{ba}$$
$$M^{p,e-q}_{ba} = (E_b - E_a)\int\prod d\vec{r}_i \Psi^*_b(\{\vec{r}_i\})\left(\sum_i (\vec{k}\cdot\vec{r}_i)(\vec{e}_{\vec{k},\sigma}\cdot\vec{r}_i)\frac{ie_i}{2\hbar}\right)\Psi_a(\{\vec{r}_i\})$$
$$M^{p,m-d_1}_{ba} = \int\prod d\vec{r}_i \Psi^*_b(\{\vec{r}_i\})\left(\sum_i \frac{i\hbar e_i}{m_i}\left(\frac{1}{2}(\vec{k}\times\vec{e}_{\vec{k}\sigma})\cdot\vec{\ell}\right)\Psi_a(\{\vec{r}_i\})\right.$$

This $M^{p,e-q}_{ba}$ is called the matrix element of the double-dipole transition. The contribution of $M^{p,m-d_1}_{ba}$ together with the contribution of (261) is called the matrix element of the magnetic dipole transition in the following. The contribution of $M^{p,m-d_2}_{ba}$ is used for handling the first-oerder contribution of $H^{(s)}$ via $e^{i\vec{k}\cdot\vec{r}} = 1$ and the dipole approximation:

$$M^{p,m-d}_{ba} = \int\prod d\vec{r}_i \Psi^*_b(\{\vec{r}_i\})\left(\sum_i \frac{i\hbar e_i}{2m_i}(\vec{e}_{\vec{k}\sigma}\times\vec{k})\cdot\vec{M}\right)\Psi_a(\{\vec{r}_i\})$$
$$\vec{M} = \vec{\ell} + \vec{\sigma} = \vec{\ell} + 2\vec{s}$$

Before we move on to demonstrate a much more simple calculation for the electric dipole approximation, let us take care of the calculation for the density of states in the radiation field. Suppose the system is in a box having side length L, the number of existing states $\rho(E)dE$ found at energy $[E, E+dE]$ then be disintegrated into the solid angle $d\Omega$ and the wave number $[k, k+dk]$:

[247] Confirm the following relation:

$$(\vec{k}\times\vec{e})(\vec{r}\times\vec{\nabla}) = \epsilon_{ijk}k_j e_k \epsilon_{iab} r_a \partial_b = (\delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka})k_j e_k r_a \partial_b$$
$$= k_j e_k r_j \partial_k - k_j e_k r_k \partial_j$$
$$[H_p, r_i r_j] = r_i[H_p, r_j] + [H_p, r_i]r_j = -\frac{\hbar^2}{m}(r_i\partial_j + \partial_j r_i)$$

And thus,

$$(\vec{k}\cdot\vec{r})(\vec{e}\cdot\vec{\nabla}) = k_i r_i e_j \partial_j = \frac{1}{2}k_i e_j(r_i\partial_j - r_j\partial_i) + \frac{1}{2}k_i e_j(r_i\partial_j + r_j\partial_i)$$
$$= \frac{1}{2}(\vec{k}\times\vec{e})\cdot(\vec{r}\times\vec{\nabla}) + \frac{1}{2}[H_p, (\vec{k}\cdot\vec{r})(\vec{e}\cdot\vec{r})]$$

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$$\rho(E) = V\frac{1}{(2\pi)^3}\frac{\omega^2}{\hbar c^3}d\Omega$$

14.3 Light Emission

We consider the following transition based on our discussion in the last subsection:

	States of atomic system	Energy of atomic system	Radiation field
Initial state	a	E_a	$\{n_i\}$
Final state	b	E_b	$^\exists\nu\ n_\nu + 1$

The energy of the emitted light can be expressed by the conservation of energy (delta function of the Fermi 's golden rule):

$$\hbar\omega = E_a - E_b$$

The emission probability of the light $wd\Omega$ into the solid angle $d\Omega$ as the polarized light σ per unit time is determined via the Fermi 's golden rule:

$$wd\Omega = \frac{2\pi}{\hbar} \times \frac{1}{\epsilon_0 V}\omega^2|\mu_\sigma^{\mathrm{T}}|^2 \times \frac{\hbar}{2\omega}(\bar{n}_{k\sigma}+1) \times \rho(E)$$

For the number of photons detected in the radiation field, $\bar{n}_{k\sigma}$ is used in the above as the average value of the wave number k and the polarized light σ. By organizing the above, we obtain:

$$\begin{aligned} w &= w_{sp} + w_{ind} = \frac{\omega^3}{8\pi^2\epsilon_0\hbar c^3}|\mu_\sigma^{\mathrm{T}}|^2(\bar{n}_{k\sigma}+1) \\ w_{sp} &= \frac{\omega^3}{8\pi^2\epsilon_0\hbar c^3}|\mu_\sigma^{\mathrm{T}}|^2\bar{n}_{k\sigma} \\ w_{ind} &= \frac{\omega^3}{8\pi^2\epsilon_0\hbar c^3}|\mu_\sigma^{\mathrm{T}}|^2 \end{aligned}$$

In the above equations, w_{ind} is proportional to $\bar{n}_{k\sigma}$, and which is known as the induced emission while the rest of the terms are known as the spontaneous emission.

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$$\begin{aligned} \rho dE &= \frac{dkk^2d\Omega}{(\frac{2\pi}{L})^3} = V\frac{k^2dkd\Omega}{(2\pi)^3} \\ E &= \hbar ck \\ \rho(E) &= V\frac{1}{(2\pi)^3}\frac{E^2}{(\hbar c)^3}d\Omega = V\frac{1}{(2\pi)^3}\frac{\omega^2}{\hbar c^3}d\Omega \end{aligned}$$

14.4 Light Absorption

The transition occurred by the light absorption can be considered in the same way we did for the emission of the light so that we can write down the following expression by use of $n_{\vec{k}\sigma} + 1 \to n_{\vec{k}\sigma}$

$$w_a = \frac{\omega^3}{8\pi^2\epsilon_0\hbar c^3}|\mu_\sigma^{\mathrm{T}}|^2\bar{n}_{k\sigma}$$

Note that this representation above can be also written in another way by letting the strength of incident light $I(\omega)d\omega$ be [249]

$$I(\omega)d\omega = c\frac{\hbar\omega n}{V}\rho_\omega d\omega = (velocity)(energydensity)\rho_\omega d\omega$$

thus,

$$w_a = \frac{\pi}{\epsilon_0\hbar^2 c}|\mu_\sigma^{\mathrm{T}}|^2 I(\omega)$$

If the two-level system a and b is thermal equilibrium through the radiation field ($E_b - E_a = \hbar\omega$), we let the atomic numbers of respective level be N_a and N_b to define the transition matrix elements for the particles system as $A_{a\to b} = A_{b\to a}$ therefore, we obtain:

$$N_b A_{b\to a}(n+1) = N_a A_{a\to b} n$$

We assume the Boltzmann distribution for the particles system:

$$\frac{N_b}{N_a} = e^{-(E_b-E_a)/k_BT} = e^{-hbar\omega/k_BT}$$

which gives the Planck 's radiation formula:

$$n = \frac{1}{e^{\hbar\omega/k_BT} - 1}$$

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$$\rho(E)dE = \tilde{\rho}(\omega)d\omega$$

which gives

$$\tilde{\rho}(\omega) = \rho(E)\hbar$$

$$I(\omega) = \frac{\hbar^2\omega cn}{V}\rho(E)$$