PartII
Relativistic Quantum Mechanics

In order to discuss the spin of an electron, the effect arising from relativity must be fully considered. In the following series of sections we sill discuss this important theory of relativity.

4 Special Relativity (Classical Theory)

First, we begin by reviewing the classical relativity theory. We use the following notation:

$$x^\mu = (x^0, x^1, x^2, x^3 = (ct, x, y, z)$$

We write the metric tensors (will be discussed later) in special relativity

$$\begin{aligned} g_{\mu\nu} &= g_{\nu\mu} = \text{diag } (1, -1, -1, -1) \\ g^{\mu\nu} &= g^{\nu\mu} = (g_{\mu\nu})^{-1} = \text{diag } (1, -1, -1, -1) \\ g_{\mu\nu} g^{\nu\rho} &= \delta_\mu{}^\rho \end{aligned}$$

The indices can be raised and lowered as below:

$$a_\mu = g_{\mu\nu} a^\nu$$

This yields

$$g^\mu{}_\nu = g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu{}_\nu$$

More generally, we can express in the notation

$$a_0 = a^0,\ a_1 = -a^1,\ a_= - a^2,\ a_3 = -a^3$$

Which gives

$$a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b} = a_0 b_0 - \vec{a} \cdot \vec{b}$$

For

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

we can write

$$\partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta = -\Box$$

4.1 Lorentz Transformation

We call the Lorentz transformation for the real linear transformations (coordinate transformations) that conserve the norm $|x|^2 = g_{\mu\nu}x^\mu x^\nu$. (We denote the coordinates of the fixed points in space time, which we measured by another frame to be x^μ, $x'^{\mu'}$.)

$$
\begin{aligned}
x'^{\mu'} &= \Omega^{\mu'}{}_\nu x^\nu \\
(\Omega^{\mu'}{}_\nu)^* &= \Omega^{\mu'}{}_\nu \\
|x'|^2 &= |x|^2 \\
g'_{\mu'\nu'}x'^{\mu'}x'^{\nu'} &= g_{\mu\nu}x^\mu x^\nu \\
g'_{\mu\nu} &= g_{\mu\nu} = \text{ diag } (1,-1,-1,-1)
\end{aligned}
$$

From which, we can derive the conditions below. [78] [79] [80]

$$
\begin{aligned}
g_{\lambda\kappa} &= g'_{\mu'\nu'}\Omega^{\mu'}{}_\lambda\Omega^{\nu'}{}_\kappa \\
\delta^\rho_\kappa &= g^\rho_\kappa = \Omega^{\mu'\rho}\Omega_{\mu'\kappa} = (\Omega_{\mu'}{}^\rho\Omega^{\mu'}{}_\kappa)
\end{aligned}
$$

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[78]

$$
\begin{aligned}
g_{\mu\nu}g^{\nu\lambda} =& \delta^\lambda_\mu \\
=& g^\lambda_\mu
\end{aligned}
$$

[79]

$$
\begin{aligned}
x'^{\mu'} &= \Omega^{\mu'}{}_\nu x^\nu \\
(\Omega^{\mu'}{}_\nu)^* &= \Omega^{\mu'}{}_\nu \\
g'{}_{\mu'\nu'}x'^{\mu'}x'^{\nu'} &= g'_{\mu'\nu'}\Omega^{\mu'}{}_\lambda x^\lambda\Omega^{\nu'}{}_\kappa x^\kappa = g_{\lambda\kappa}x^\lambda x^\kappa gives \\
g_{\lambda\kappa} &= g'_{\mu'\nu'}\Omega^{\mu'}{}_\lambda\Omega^{\nu'}{}_\kappa \\
Thus, \delta^\rho{}_\kappa &= g^{\rho\lambda}g_{\lambda\kappa} \\
&= g^{\rho\lambda}g'_{\mu'\nu'}\Omega^{\mu'}{}_\lambda\Omega^{\nu'}{}_\kappa \\
&= \Omega^{\mu'\rho}\Omega_{\mu'\kappa}
\end{aligned}
$$

[80] For the arbitrary quantities X, Y, we write

$$
X^\mu Y_\mu = X_\kappa g^{\kappa\mu}Y^\lambda g_{\lambda\mu} = X_\kappa Y^\lambda g^\kappa{}_\lambda = X_\kappa Y^\kappa
$$

The inverse transformation can be written [81]

$$x'^{\mu}\Omega_{\mu}{}^{\kappa} = x^{\kappa}$$

The following relation is also valid: [82]

$$\begin{aligned}\Omega_{\nu\kappa}\Omega^{\rho\kappa} &= \delta_{\nu}^{\rho} \\ &= \Omega_{\nu}{}^{\kappa}\Omega^{\rho}{}_{\kappa} = g_{\nu}^{\rho}\end{aligned}$$

All together, we can express [83]

$$\begin{aligned}(\Omega^{-1})^{\mu}{}_{\nu} &= (\Omega)_{\nu}{}^{\mu} \\ (\Omega^{-1})_{\mu}{}^{\nu} &= \Omega^{\nu}{}_{\mu} \\ (\Omega^{-1})_{\mu\nu} &= \Omega^{\nu\mu} \\ (\boldsymbol{\Omega})_{\mu\nu} &\equiv \Omega_{\mu\nu} \text{として} \\ \tilde{\boldsymbol{\Omega}}\boldsymbol{\Omega} = \boldsymbol{\Omega}\tilde{\boldsymbol{\Omega}} &= \boldsymbol{I}\end{aligned}$$

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$$\begin{aligned}x'^{\mu}g_{\mu\rho} &= g_{\rho\mu}\Omega^{\mu}{}_{\nu}x^{\nu} = \Omega_{\rho\nu}x^{\nu} \\ x'^{\mu}g_{\mu\rho}\Omega^{\rho\kappa} &= \Omega^{\rho\kappa}\Omega_{\rho\nu}x^{\nu} \\ x'^{\mu}\Omega_{\mu}{}^{\kappa} &= \delta_{\nu}^{\kappa}x^{\nu} = x^{\kappa}\end{aligned}$$

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$$\begin{aligned}g_{\rho\kappa}x^{\rho}x^{\kappa} &= g_{\rho\kappa}x'^{\nu}\Omega_{\nu}{}^{\rho}x'^{\mu}\Omega_{\mu}{}^{\kappa} = g_{\nu\mu}x'^{\nu}x'^{\mu} \\ g_{\rho\kappa}\Omega_{\nu}{}^{\rho}\Omega_{\mu}{}^{\kappa} &= \Omega_{\nu\kappa}\Omega_{\mu}{}^{\kappa} = g_{\nu\mu} \\ \Omega_{\nu\kappa}\Omega^{\rho\kappa} &= g_{\nu\mu}g^{\mu\rho} = \delta_{\nu}^{\rho}\end{aligned}$$

[83]Let us put

$$(\Omega^{-1})^{\mu}{}_{\nu} = (\Omega)_{\nu}{}^{\mu}$$

This gives

$$\begin{aligned}(\Omega^{-1})^{\mu}{}_{\nu}\Omega^{\nu}{}_{\kappa} &= (\Omega)_{\nu}{}^{\mu}\Omega^{\nu}{}_{\kappa} = \delta_{\kappa}^{\mu} \\ \Omega^{\mu}{}_{\nu}(\Omega^{-1})^{\nu}{}_{\kappa} &= \Omega^{\mu}{}_{\nu}\Omega_{\kappa}{}^{\nu} = \delta_{\kappa}^{\mu}\end{aligned}$$

and furthuer we can write

$$(\Omega^{-1})_{\mu}{}^{\nu} = ((\Omega^{-1})^{-1})^{\nu}{}_{\mu} = \Omega^{\nu}{}_{\mu}$$

The Example pf the Lorentz Transformation

- Rotation *phi* around $z-$ axis

$$\begin{pmatrix} ct'^0 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct^0 \\ x \\ y \\ z \end{pmatrix}$$

- Special Lorentz transformation with velocity $v = c\tanh\phi$ in direction of $x-$ axis: [84]

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Tensor

Under the coordinate transformation $x \to x'$, the physical quantity $\mathcal{O}(P)$ in space time p, which follows the transformations described below are called in each name below. (A point in space time $P(\{x^\mu\})$ defined by a coordinate system is $\{x^\mu\}$, while it is defined as $P(\{x'^\mu\})$ by another coordinate system of $'$. This gives the functional relationship $x'^\mu = x'^\mu(\{x^\nu\})$.)

$$\begin{aligned} \frac{\partial x'^{\mu'}}{\partial x^\nu} &\equiv x^{\mu'}_{,\nu} = \Omega^{\mu'}{}_\nu \\ \frac{\partial x^\nu}{\partial x'^{\mu'}} &\equiv x^\nu_{,\mu'} = \Omega_{\mu'}{}^\nu \\ x^{\mu'}_{,\nu} x^\nu_{,\kappa'} &= \frac{\partial x'^{\mu'}}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^{\kappa'}} = \delta^{\mu'}{}_{\kappa'} \\ x^\mu_{,\nu'} x^{\nu'}_{,\kappa} &= \frac{\partial x^\mu}{\partial x'^{\nu'}} \frac{\partial x'^{\nu'}}{\partial x^\kappa} = \delta^\mu{}_\kappa \end{aligned}$$

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[84]FOr this we let $x = 0$ and write

$$\begin{aligned} t' &= t\cosh\phi, \quad x' = -ct\sinh\phi \\ \frac{x'}{t'} &= -c\tanh\phi \end{aligned}$$

This above implies that the system x' is i uniform motion with the velocity $-c\tanh\phi$ to the system x

- Scalar

$$T' = T$$

- Contravariant vector

$$T'^{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^{\nu}} T^{\nu} = x^{\mu'}_{,\nu} T^{\nu} = \Omega^{\mu'}{}_{\nu} T^{\nu}$$

- Covariant vector

$$T'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} T_{\nu} = \Omega_{\mu}{}^{\nu} T_{\nu}$$

- Contravariant of the 1st order and the 2nd order (examples)

$$T'^{\mu_1}{}_{\kappa_1\kappa_2} = \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \frac{\partial x^{\rho_1}}{\partial x'^{\kappa_1}} \frac{\partial x^{\rho_2}}{\partial x'^{\kappa_2}} T^{\nu_1}{}_{\rho_1\rho_2} = \Omega^{\mu_1}{}_{\nu_1} \Omega_{\kappa_1}{}^{\rho_1} \Omega_{\kappa_2}{}^{\rho_2} T'^{\nu_1}{}_{\rho_1\rho_2}$$

- The contraction $A^{\mu} B_{\mu}$, for example, of the contravariant vector and covariant vector is the scalar. [85]
- What contracts with the contravariant vector to become a scalar is called the covariant vector.
- The second order covariant tensor is $g_{\mu\nu}$. [86]

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[85]

$$A'^{\mu} B'_{\mu} = \Omega^{\mu}{}_{\nu} A^{\nu} \Omega_{\mu}{}^{\kappa} B_{\kappa} = \Omega^{\mu}{}_{\nu} \Omega_{\mu}{}^{\kappa} A^{\nu} B_{\kappa} = g^{\mu\rho} \Omega_{\rho_\nu} g_{\mu\eta} \Omega^{\eta\kappa} A^{\nu} B_{\kappa} = \Omega_{\rho_\nu} \Omega^{\rho\kappa} A^{\nu} B_{\kappa} = A^{\nu} B_{\nu}$$

[86]

$$\begin{aligned} ds'^2 &= g'_{\mu\nu} dx'^{\mu} dx'^{\nu} = g'_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\rho}} dx^{\rho} \frac{\partial x'^{\nu}}{\partial x^{\kappa}} dx^{\kappa} \\ ds^2 &= g_{\rho\kappa} dx^{\rho} dx^{\kappa} \end{aligned}$$

giving $ds = ds'$ thus,

$$\begin{aligned} g'_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\kappa}} &= g_{\rho\kappa} \\ g'_{\mu\nu} &= \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\kappa}}{\partial x'^{\nu}} g_{\rho\kappa} \end{aligned}$$

4.2 Effects of Free Particles

The action integral is defined as:

$$\begin{aligned} S &= -mc\int_a^b ds = \int_{t_a}^{t_b} L\,dt \\ ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ L &= -mc\sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = -mc^2\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}, \quad \vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \end{aligned}$$

The Lorentz transformations $x'^\mu = \Omega^\mu{}_\nu x^\nu$ gives ($g' = g$) , and the line element stays invariant $ds = ds'$. This fact implies that the action is being interpreted as Lorentz invariant.

In the non-relativity limit:

$$L \;\rightarrow\; -mc^2(1-\frac{1}{2}\frac{v^2}{c^2}) = -mc^2 + \frac{1}{2}mv^2$$

where the kinetic energy is indeed being given, while excluding the constant values in the limit. The momentum can be written

$$\begin{aligned} \vec{p} &= \frac{\partial L}{\partial \dot{\vec{r}}} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}} \equiv M\vec{v} \\ M &= \frac{m}{\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}} \end{aligned}$$

and let M be the relative mass. The Hamiltonian H and the energy E can be defined as:

$$\begin{aligned} H &= E = \vec{p}\cdot\dot{\vec{r}} - L = \vec{p}\cdot\vec{v} - L \\ &= \frac{mv^2}{\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}} + mc^2\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}} = \frac{mc^2}{\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}} = Mc^2 \end{aligned}$$

Therefore, in the non-relativity limit, we have

$$E \;\rightarrow\; mc^2(1+\frac{1}{2}\frac{v^2}{c^2}) = mc^2 + \frac{1}{2}mv^2$$

which naturally gives the rest energy mc^2 . The following relations can be derived between the energy and the momentum: [87]

$$
\begin{aligned}
c\vec{p} &= \frac{\vec{v}}{c}E \\
H &= E = c\sqrt{p^2 + m^2c^2}
\end{aligned}
$$

Especially where super-relativistic $v \approx c$, [88] the relation with $E \approx cp$ particularly with light can be

$$E = cp$$

The canonical equation can be written [89]

$$
\begin{aligned}
\dot{\vec{r}} &= \vec{v} = \frac{\partial H}{\partial \vec{p}} = \frac{c^2\vec{p}}{E} \\
\dot{\vec{p}} &= -\frac{\partial H}{\partial \vec{r}} = 0
\end{aligned}
$$

which giving $\vec{p} = \frac{E\vec{v}}{c^2} = M\vec{v}$ by the first equation, we may make a substitution into the second equation to write

$$\dot{\vec{p}} = \frac{d(M\vec{v})}{dt} = \frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = 0$$

[87] We can use

$$\vec{p} = \vec{v}\frac{E}{c^2}$$

to cancel v from the energy equation such that

$$
\begin{aligned}
E^2(1-\frac{v^2}{c^2}) &= m^2c^4 \\
E^2(1-c^2\frac{p^2}{E^2}) &= m^2c^4 \\
E^2 &= m^2c^4 + c^2p^2
\end{aligned}
$$

[88]

$$
\begin{aligned}
\frac{p^2}{E} &\approx \frac{m}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} = \frac{E}{c^2} \\
E &\approx cp
\end{aligned}
$$

[89]

$$\dot{\vec{r}} = \vec{v} = \frac{\partial H}{\partial \vec{p}} = c\frac{2\vec{p}}{2\sqrt{p^2+m^2c^2}} = \frac{c^2\vec{p}}{E}$$

This in fact is an equation of motion.

To discuss the Lorentz invariance in more explicit form, we can use the variation principle to write the differential of the curve 's parameter τ with $'$. Rewrite the action of the curve with common parameter τ, and write the Lagrangian of the common parameter as L $(S = \int_{\tau_a}^{\tau_b} L\,d\tau$). [90] Thus,

$$\frac{\delta L}{\delta x^\mu} = \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial x^{\mu\prime}} = -mc\frac{d}{d\tau}\left(\frac{g_{\mu\nu}x^{\nu\prime}}{\sqrt{g_{\rho\kappa}x^{\rho\prime}x^{\kappa\prime}}}\right) = 0$$

We take parameter τ as $ds = cd\tau$,($x^{\mu\prime}x_\mu{}' = c^2$) that gives (proper time) [91] [92]

$$\frac{d^2x^\kappa}{d\tau^2} = 0$$

From this, we can now consider the free-particle. If we have $\tau = t$, the relational expression for the components of $\mu = 0$ can be written [93]

$$\frac{d}{dt}\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{dE}{dt} = 0$$

indicating that the energy is being conserved. The conservation of momentum can

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[90]

$$\begin{aligned} L &= -mc\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = -mc\sqrt{g_{\mu\nu}x^{\mu\prime}x^{\nu\prime}} \\ \frac{\delta L}{\delta x^\mu} &= \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial x^{\mu\prime}} = -mc\frac{d}{d\tau}\left(\frac{g_{\mu\nu}x^{\nu\prime}}{\sqrt{\quad}}\right) = 0 \end{aligned}$$

[91] Let parameter τ be $ds = cd\tau$,($x^{\mu\prime}x_\mu{}' = c^2$) and write

$$\begin{aligned} s = \int_{s_a}^{s} ds = \int_{\tau_a}^{\tau}\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau = \int_{\tau_a}^{\tau}\sqrt{x^{\mu\prime}x_\nu{}'}d\tau \\ ds = \sqrt{x^{\mu\prime}x_\nu{}'}d\tau = cd\tau \\ x^{\mu\prime}x_\nu{}' = c^2 \end{aligned}$$

[92]

$$g^{\kappa\mu}\frac{\delta L}{\delta x^\mu} = = -mcg^{\kappa\mu}\frac{d}{d\tau}g_{\mu\nu}x^{\nu\prime} = -mc\delta^\kappa{}_\nu\frac{d^2x^\nu}{d\tau^2} = -mc\frac{d^2x^\kappa}{d\tau^2} = 0$$

[93]

$$\frac{d}{dt}\frac{c}{c\sqrt{1-\frac{v^2}{c^2}}} = 0$$

be given by $\mu = i = 1,2,3$: [94]

$$\frac{d}{dt}\frac{mc\dot{\vec{r}}}{\sqrt{1-\frac{v^2}{c^2}}} = \vec{0}$$

When we let four-momentum be $p_\mu = \dfrac{\partial L}{\partial \dot{x}^\mu}$ as we will cover it in our next section, we have [95]

$$\begin{aligned} p_0 &= -Mc = -\frac{E}{c} \\ p_i &= p_{x,y,z} = \left(\frac{m\dot{\vec{r}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)_i \end{aligned}$$

which giving the covariance of vectors for the Lorentz transform.

4.3 Particle Motion in Electromagnetic Field (Lagrange Formulation)

Let us describe below as the action integral:

$$\begin{aligned} S &= S_0 + S_{el} \\ S_0 &= -mc\int_a^b ds = -mc\int_{\tau_a}^{\tau_b} d\tau\,\sqrt{g_{\mu\nu}x^{\mu\prime}x^{\nu\prime}} = \int_{t_a}^{t_b} dt\,L_0 \\ L_0 &= -mc\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \\ S_{el} &= -e\int A_\mu dx^\mu = -e\int_{\tau_a}^{\tau_b} A_\kappa x^{\kappa\prime}d\tau = \int_{t_a}^{t_b} dt\,L_{el} \\ L_{el} &= -eA_\mu\frac{dx^\mu}{dt} = -e\phi + e\dot{\vec{r}}\cdot\vec{A} \end{aligned}$$

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$$\frac{d}{dt}\frac{-\dot{x}^\mu}{c\sqrt{1-\frac{v^2}{c^2}}} = 0$$

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$$\begin{aligned} p_\mu &= -mc\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \\ p_0 &= -mc\frac{c}{c\sqrt{1-\frac{v^2}{c^2}}} = -Mc = -\frac{E}{c} \\ p_i &= p_{x,y,z} = -mc\frac{-\dot{x}^i}{c\sqrt{1-\frac{v^2}{c^2}}} = \left(\frac{m\dot{\vec{r}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)_i \end{aligned}$$

The four-vector potential can be written

$$
\begin{aligned}
A_0 &= A^0 = \frac{1}{c}\phi \\
A_i &= -A^i, \quad A^1 = A_x, A^2 = A_y, A^3 = A_z
\end{aligned}
$$

Where $\dot{x}^\mu = \dfrac{dx^\mu}{dt}$ is the four-velocity.

Note that the Lorentz invariance of this action is obeyed by the covariant vector A_μ. The covariance of A_μ is obeyed by the observation given by the Maxwell 's equation as well as by the conservation of the electric charge. [96]

In those actions, we use the variation principle in which the equation of motion

[96]The covariance of A_μ is obeyed by the observation because of the Maxwell 's equation and the conservation of the electric charge. From our later discussion, the Maxwell 's equation can be defined by $\vec{B} = \mathrm{div}\,\vec{A}, \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$, which are equivalent to the two equations below:

$$\begin{aligned}
\Box \vec{A} &= \Delta \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \vec{\nabla}(\mathrm{div}\,\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}) - \mu_0 \vec{j} \\
\Delta \phi &= -\frac{\partial}{\partial t}\mathrm{div}\,\vec{A} - \frac{\rho}{\epsilon_0}
\end{aligned}$$

Under a condition called the Lorentz (gauge) condition

$$\mathrm{div}\,\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = \frac{\partial A^\mu}{\partial x^\mu} = \partial_\mu A^\mu = 0$$

The two equivalent equations we described above can be written

$$\begin{aligned}
\Box \vec{A} &= -\mu_0 \vec{j} \\
\Box \phi &= -c^2 \mu_0 \rho
\end{aligned}$$

Here we let the four-current j^μ be

$$j_0 = c\rho,\ j^1 = j_x,\ j^2 = j_y,\ j^3 = j_z$$

which giving the Maxwell 's equation

$$\Box A^\mu = -\mu_0 j^\mu$$

.

For the conservation of electric charge

$$0 = \mathrm{div}\,\vec{j} + \frac{\partial \rho}{\partial t} = \partial_\mu j^\mu$$

which is (experimentally) understood to be the Lorentz invariant. This gives the contravariant vector j^μ and A^μ. Note that the Lorentz condition $\partial_\mu A^\mu = 0$ in fact expresses the relation for the scalar, and remains invariant to the Lorentz transformation,

$$\Box \partial_\mu A^\mu = -\mu_0 \partial_\mu j^\mu = 0$$

This is compatible with the field equation. Now, the gauge transformation

$$\begin{aligned}
\vec{A} \to \bar{\vec{A}} = \vec{A} + \vec{\nabla}\chi \\
\phi \to \bar{\phi} = \phi - \frac{\partial \chi}{\partial t}
\end{aligned}$$

can be written

$$A_\mu \to \bar{A}_\mu = A_\mu + \partial_\mu \chi$$

To write $\vec{E},\ \vec{B}$ in four-form, we let the second order covariant tensor be

$$f_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = -f_{\nu\mu}$$

We may write down

$$\begin{aligned}
f_{01} &= \partial_0 A_1 - \partial_1 A_0 = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial x} = \frac{1}{c}E_x \\
f_{02} &= \frac{1}{c}E_y \\
f_{03} &= \frac{1}{c}E_z \\
f_{12} &= \partial_1 A_2 - \partial_2 A_1 = -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -B_z \\
f_{13} &= \partial_1 A_3 - \partial_3 A_1 = -\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z} = B_y \\
f_{23} &= \partial_2 A_3 - \partial_3 A_2 = -\frac{\partial A_z}{\partial y} + \frac{\partial A_y}{\partial z} = -B_x
\end{aligned}$$

Organize the above and rewrite

$$f_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

These indeed stay invariant under gauge transformation:

$$\begin{aligned}
\bar{f}_{\mu\nu} &= \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu \\
&= \partial_\mu (A_\nu + \partial_\nu \chi) - \partial_\nu (A_\mu + \partial_\mu \chi) = f_{\mu\nu}
\end{aligned}$$

[97] takes the proper-time parameter. In the Lorentz invariant form, we can write

$$\begin{aligned} m\frac{d^2x^\rho}{d\tau^2} &= -e\frac{dx_\nu}{d\tau}f^{\nu\rho} \\ f_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & 0 & B_x & 0 \end{pmatrix} \end{aligned}$$

Rewrite the above as

$$m\frac{d^2x^\rho}{d\tau^2} = F^\rho$$
$$F^\rho = -e\frac{dx_\nu}{d\tau}f^{\nu\rho}$$

We call F^ρ the four-force. This equation of motion embodies the all four forces

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$$\begin{aligned} \frac{\delta L_0}{\delta x^\mu} &= mc\frac{d}{d\tau}\left(\frac{g_{\mu\nu}x^{\nu\prime}}{\sqrt{g_{\alpha\beta}x^{\alpha\prime}x^{\beta\prime}}}\right) \\ \frac{\delta L_{el}}{\delta x^\mu} &= -e\big(\partial_\mu(A_\kappa x^{\kappa\prime}) - \frac{dA_\mu}{d\tau}\big) \\ &= -e\big(\partial_\mu(A_\kappa x^{\kappa\prime}) - x^{\nu\prime}\partial_\nu A_\mu\big) \\ &= -e\big(x^{\kappa\prime}\partial_\mu A_\kappa - x^{\nu\prime}\partial_\nu A_\mu\big) \\ &= -e(\partial_\mu A_\nu - \partial_\nu A_\mu)x^{\nu\prime} = -f_{\mu\nu}x^{\nu\prime} = f_{\nu\mu}x^{\nu\prime} \end{aligned}$$

In which we take proper-time parameter, and written

$$\begin{aligned} mg_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + e\frac{dx^\nu}{d\tau}f_{\nu\mu} &= 0 \\ mg^{\rho\mu}g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} &= -eg^{\rho\mu}\frac{dx^\nu}{d\tau}f_{\nu\mu} \\ m\frac{d^2x^\rho}{d\tau^2} &= -e\frac{dx^\nu}{d\tau}f_\nu{}^\rho = -e\frac{dx_\nu}{d\tau}f^{\nu\rho} \end{aligned}$$

We define $\tau = t$, so that

$$\begin{aligned} m\frac{d}{dt}\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{1-\frac{v^2}{c^2}}} &= e\dot{x}^\kappa f_{\mu\kappa} \\ \frac{d\pi_\mu}{dt} &= e\dot{x}^\kappa f_{\mu\kappa}, \quad (\pi_\mu takesthetimetforthecommonparameter\tau; thatiswhen\tau = t) \\ \pi_\mu &= m\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{1-\frac{v^2}{c^2}}} = Mg_{\mu\nu}\dot{x}^\nu = M\dot{x}_\mu \\ \pi_\mu\pi^\mu &= \frac{m^2\dot{x}_\mu\dot{x}^\mu}{1-\frac{v^2}{c^2}} = \frac{m^2(c^2-v^2)}{1-\frac{v^2}{c^2}} = m^2c^2 \end{aligned}$$

that are not independent but has a linear relation among them: [98]

$$u_\mu F^\nu = 0$$
$$u_\mu = \frac{dx_\mu}{d\tau}$$

Where u^μ is the four-velocity, we can write

$$u^\mu u_\mu = c^2$$
$$u^\mu = (c\frac{dt}{d\tau}, \vec{v}\frac{dt}{d\tau})$$

[99]

With time t, this equation of motion can be written [100]

$$\begin{aligned} \frac{d\pi_\mu}{dt} &= e\dot{x}^\kappa f_{\mu\kappa} \\ \pi_\mu &= m\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{1-\frac{v^2}{c^2}}} = M\dot{x}_\mu = mu_\mu \\ \pi_\mu \pi^\mu &= m^2c^2 \end{aligned}$$

98

$$\frac{dx_\mu}{d\tau}F^\mu = -e\frac{dx_\mu}{d\tau}\frac{dx_\kappa}{d\tau}f^{\kappa\mu} = 0; antisymmetric of \quad f^{\kappa\mu}$$

99 We can rewrite

$$u^0 = c\frac{dt}{d\tau}$$
$$u^i = \frac{dx^i}{d\tau}$$

$$\sqrt{(\frac{dt}{d\tau})^2 - \frac{1}{c^2}(\frac{dt}{d\tau}\dot{\vec{r}})^2} = 1$$
$$dt\sqrt{1-\frac{v^2}{c^2}} = d\tau$$
$$M\dot{x}^\mu = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\dot{x}^\mu = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dx^\mu}{dt} = m\frac{dx^\mu}{d\tau} = mu^\mu$$

100

$$\pi_\mu\pi^\mu = M^2c^2 - \vec{\pi}^2 = m^2c^2$$
$$\vec{\pi} = M\vec{v} = (\pi^1, \pi^2, \pi^3) = (-\pi_1, -\pi_2, -\pi_3)$$

For each component we can write in the forms [101]

$$\begin{aligned}\frac{d(Mc^2)}{dt} &= e\vec{v}\cdot\vec{E}\\ \frac{d\vec{\pi}}{dt} &= e(\vec{E}+\vec{v}\times\vec{B})\\ M^2c^2-\vec{\pi}^2 &= m^2c^2\end{aligned}$$

Now let us rewrite the above:

$$\begin{aligned}M &= \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\\ \vec{\pi} &= M\vec{v} = (\pi^1,\pi^2,\pi^3) = (-\pi_1,-\pi_2,-\pi_3)\\ \vec{v} &= \dot{\vec{r}}\end{aligned}$$

We can also confirm the equation

$$\vec{v}\cdot\frac{d\pi}{dt} = e\vec{E}\cdot\vec{v} = \frac{dMc^2}{dt}$$

[101]

$$\begin{aligned}\pi_0 &= \frac{mc\cdot 1}{\sqrt{1-\frac{v^2}{c^2}}} = Mc\\ \pi_1 &= \frac{-m\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}} = -M\dot{x} = -\pi^1 = -\pi_x\\ \pi_2 &= -M\dot{y}-\pi^2 = -\pi_y, \quad \pi_3 = -M\dot{z} = -\pi^3 = -\pi_z\end{aligned}$$

The zeroth component gives

$$\begin{aligned}\frac{d\pi_0}{dt} &= c\frac{dM}{dt} = e\dot{x}^\kappa f_{0\kappa} = \frac{e}{c}(\dot{x}E_x+\dot{y}E_y+\dot{z}E_z)\\ \frac{d(Mc^2)}{dt} &= e\vec{v}\cdot\vec{E}\end{aligned}$$

While the first component gives

$$\begin{aligned}\frac{d\pi_1}{dt} &= -\frac{d(M\dot{x})}{dt} = e\dot{x}^\kappa f_{1\kappa} = e\Big(-\frac{c}{c}E_x-\dot{y}B_z+\dot{z}B_y\Big)\\ \frac{d\pi_x}{dt} &= e(\vec{E}+\dot{\vec{r}}\times\vec{B})_x\\ Lilewisewewrite \qquad & \frac{d\pi_y}{dt} = e(\vec{E}+\dot{\vec{r}}\times\vec{B})_y, \quad \frac{d\pi_z}{dt} = e(\vec{E}+\dot{\vec{r}}\times\vec{B})_z\end{aligned}$$

Thus, only $\frac{d\vec{\pi}}{dt} = e(\vec{E} + \vec{v} \times \vec{B})$ remains independent for the equation of motion. [102]

4.4 Particle Motion in Electromagnetic Field (Hamilton Formulation)

We now discuss the particle motion in electromagnetic field by Hamiltonian forms. Where $(\tau = t)$, the canonical momentum is defined as [103]

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial \dot{x}^\mu} \\ &= -\frac{m\dot{x}_\mu}{\sqrt{1-\frac{v^2}{c^2}}} - eA_\mu = -M\dot{x}_\mu - eA_\mu \end{aligned}$$

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$$\begin{aligned} \vec{v}\cdot\frac{d\vec{\pi}}{dt} &= e\vec{E}\cdot\vec{v} \\ &= \frac{dM}{dt}\vec{v}^2 + M\vec{v}\cdot\frac{d\vec{v}}{dt} \\ &= \frac{dM}{dt}v^2 + \frac{1}{2}M\frac{dv^2}{dt} \\ &= m\frac{\frac{d}{dt}\frac{v^2}{c^2}}{2(1-\frac{v^2}{c^2})^{3/2}}v^2 + \frac{1}{2}M\frac{dv^2}{dt} \\ &= m\frac{v^2+(1-\frac{v^2}{c^2})c^2}{2(1-\frac{v^2}{c^2})^{3/2}}\frac{d}{dt}\frac{v^2}{c^2} = \frac{m}{2(1-\frac{v^2}{c^2})^{3/2}}\frac{dv^2}{dt} = \frac{dMc^2}{dt} \end{aligned}$$

$$\vec{v}\cdot\frac{d\pi}{dt} = e\vec{E}\cdot\vec{v} = \frac{dMc^2}{dt}$$

103

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial \dot{x}^\mu} \\ &= -mc\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} - eA_\mu \\ &= -\frac{m\dot{x}_\mu}{\sqrt{1-\frac{v^2}{c^2}}} - eA_\mu = -M\dot{x}_\mu - eA_\mu \\ &= -\pi_\mu - eA_\mu \end{aligned}$$

$$\begin{aligned} &p_0 = -Mc - eA_0 = -Mc - \frac{e}{c}\phi \\ &p_1 = -M\dot{x}_1 - eA_1 = +M\dot{x} + eA_x \\ &p_2 = -M\dot{x}_2 - eA_2 = +M\dot{y} + eA_y \\ &p_3 = -M\dot{x}_3 - eA_3 = +M\dot{z} + eA_z \end{aligned}$$

Each component can be written in the forms:

$$p_0 = -Mc - \frac{e}{c}\phi$$
$$p_1 = M\dot{x} + eA_x \equiv p_x$$
$$p_2 = M\dot{y} + eA_y \equiv p_y$$
$$p_3 = M\dot{z} + eA_z \equiv p_z$$

$$\vec{p} = \vec{\pi} + e\vec{A}$$

The Hamiltonian H can be defined as [104] [105]

$$\begin{aligned} H &= \sum_{\mu=1,2,3} p_\mu \dot{x}^\mu - L \\ &= c\sqrt{\vec{\pi}^2 + m^2c^2} + e\phi \\ &= c\sqrt{(\vec{p} - e\vec{A})^2 + m^2c^2} + e\phi \end{aligned}$$

[104] Giving

$$\begin{aligned} \vec{\pi} &= M\vec{v} = \vec{p} - e\vec{A} \\ M^2c^2 &= \vec{\pi}^2 + m^2c^2 \end{aligned}$$

We may write

$$Mc = \sqrt{\vec{\pi}^2 + m^2c^2}$$

More precisely we can write

$$\begin{aligned} (\vec{p} - e\vec{A})^2 &= M^2v^2 \\ (\vec{p} - e\vec{A})^2 + m^2c^2 &= \frac{m^2v^2}{1 - \frac{v^2}{c^2}} + m^2c^2 = m^2\frac{v^2 + c^2(1 - \frac{v^2}{c^2})}{1 - \frac{v^2}{c^2}} \\ &= m^2\frac{c^2}{1 - \frac{v^2}{c^2}} \\ \sqrt{(\vec{p} - e\vec{A})^2 + m^2c^2} &= \frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}} = Mc \end{aligned}$$

Thus,

$$Mc^2 = c\sqrt{(\vec{p} - e\vec{A})^2 + m^2c^2}$$

[105]

$$\begin{aligned} H &= \sum_{\mu=1,2,3} p_\mu \dot{x}^\mu - L \\ &= \sum_{\mu=0,1,2,3} p_\mu \dot{x}^\mu - p_0\dot{x}^0 - L \\ &= -p_0\dot{x}^0 + p_\mu \dot{x}^\mu - L \\ &= -p_0\dot{x}^0 - M\dot{x}_\mu\dot{x}^\mu - eA_\mu\dot{x}^\mu - (-mc^2\sqrt{1 - \frac{v^2}{c^2}} - eA_\mu\dot{x}^\mu) \\ &= -p_0\dot{x}^0 - M(c^2 - v^2) + mc^2\sqrt{1 - \frac{v^2}{c^2}} \\ &= -p_0\dot{x}^0 = Mc^2 + e\phi \\ &= c\sqrt{\vec{\pi}^2 + m^2c^2} + e\phi \\ &= c\sqrt{(\vec{p} - e\vec{A})^2 + m^2c^2} + e\phi \end{aligned}$$

In the non-relativistic limit $\frac{(\vec{p}-e\vec{A})^2}{2m} << mc^2$ the Hamiltonian can be defined as [106]

$$H \approx mc^2 + \frac{(\vec{p}-e\vec{A})^2}{2m} + e\phi$$

Recall our initial discussion is to formulate a Hamiltonian description of particle motion in electromagnetic field. The canonical equation can be given [107]

$$\begin{aligned} \vec{v} &\equiv \dot{\vec{r}} = \frac{\vec{\pi}}{M} \\ \dot{\vec{p}} &= e\vec{\nabla}(\vec{A}\cdot\vec{v} - \phi) \end{aligned}$$

Now given $\vec{p} = \vec{\pi} + e\vec{A}$, the canonical equation we described above may give the equation of motion which we described earlier:

---

[106]

$$\begin{aligned} H =& mc^2\sqrt{1 + \frac{1}{m^2c^2}(\vec{p}-e\vec{A})^2} + e\phi \\ \approx& mc^2(1 + \frac{1}{2}(\vec{p}-e\vec{A})^2) + e\phi = mc^2 + \frac{(\vec{p}-e\vec{A})^2}{2m} + e\phi \end{aligned}$$

[107] The canonical equations are written

$$\begin{aligned} \dot{\vec{r}} &= \frac{\partial H}{\partial \vec{p}} \\ \dot{\vec{p}} &= -\frac{\partial H}{\partial \vec{r}} \end{aligned}$$

we direct our attention to the first equation of $Mc = \sqrt{\vec{\pi}^2 + m^2c^2}, \vec{\pi} = \vec{p} - e\vec{A} = M\vec{v}$ and write

$$\begin{aligned} \vec{v} &\equiv \dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} \\ &= c\frac{\vec{\pi}}{\sqrt{\vec{\pi}^2 + m^2c^2}} \\ &= \frac{\vec{\pi}}{M} \end{aligned}$$

Given $\vec{\pi} = M\vec{v}$, te second equation is written

$$\begin{aligned} \dot{\vec{p}} &= -\frac{\partial H}{\partial \vec{r}} \\ &= c\frac{e\vec{\nabla}(\vec{\pi}\cdot\vec{A})}{\sqrt{\vec{\pi}^2 + m^2c^2}} - e\vec{\nabla}\phi, \ (Note\ that\ \vec{\nabla}\ does\ not\ differentiate\ \pi.) \\ &= e(\vec{\nabla}(\vec{A}\cdot\vec{v}) - \vec{\nabla}\phi), \ (Note\ that\ \vec{\nabla}\ does\ not\ differentiate\ v\ as\ we\ express\ normally.) \\ &= e\vec{\nabla}(\vec{A}\cdot\vec{v} - \phi) \end{aligned}$$

$$
\begin{aligned}
\frac{d\vec{\pi}}{dt} &= e\vec{\nabla}(\vec{A}\cdot\vec{v}-\phi) - e\frac{d\vec{A}}{dt} \\
&= e\vec{\nabla}(\vec{A}\cdot\vec{v}-\phi) - e\frac{\partial\vec{A}}{\partial t} - e(\vec{v}\cdot\vec{\nabla})\vec{A} \\
&\qquad (Note\ that\ \vec{\nabla}\ does\ not\ differentiate\ v\ \ :a\ normal\ way\ of\ expression.) \\
&= e\vec{E} + e\big(\vec{\nabla}(\vec{A}\cdot\vec{v}) - (\vec{v}\cdot\vec{\nabla})\vec{A}\big) \\
&= e(\vec{E} + \vec{v}\times\vec{B})
\end{aligned}
$$

Thus, the non-relativistic limit of the above equation speaks for itself. [108]

[108] We consider v is being independent of $\vec{r}$, and given that we have $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - (\vec{A}\cdot\vec{B})\vec{C}$ we can write

$$
\begin{aligned}
\vec{v}\times \mathrm{rot}\,\vec{A} &= \vec{v}\times(\vec{\nabla}\times\vec{A}) \\
&= \vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}
\end{aligned}
$$

or

$$
\begin{aligned}
(\vec{v}\times \mathrm{rot}\,\vec{A})_i &= \epsilon_{ijk}v_j\epsilon_{klm}\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})v_j\partial_l A_m \\
&= v_j\partial_i A_j - v_j\partial_j A_i
\end{aligned}
$$

5 Dirac Equations

5.1 Deriving the Dirac Equation

Based on the relativistic Hamiltonian we obtained in the previous section, we continue the procedures of quantization. We first write the classical Hamiltonian

$$H_{cl} \;=\; c\sqrt{(\vec{p}-e\vec{A})^2+m^2c^2}+e\phi$$

to which we make replacement $\vec{p} \to -i\hbar\vec{\nabla}$ and consider the quantum Hamiltonian. Knowing that the root sign included in above equation being somehow difficult, we may write

$$H_{D,cl} \;=\; c\vec{\alpha}\cdot(\vec{p}-e\vec{A})+\beta mc^2+e\phi$$

and use the formal equation of

$$H_{cl} \;=\; H_{D,cl}$$

from which we try determining the Hamiltonian $H_{D,cl}$ that includes no root signs. To explain further, we would like to determine the expansion coefficients $\vec{\alpha}$ and β which satisfy

$$c^2\left\{(\vec{p}-e\vec{A})^2+m^2c^2\right\} \;=\; \left\{c\vec{\alpha}\cdot(\vec{p}-e\vec{A})+\beta mc^2\right\}^2$$

To obtain such coefficients we need to have

$$\begin{aligned}
\alpha_x^2=\alpha_y^2=\alpha_z^2=\beta^2 &= 1\\
\{\alpha_i,\alpha_j\} &= \alpha_i\alpha_j+\alpha_j\alpha_i=0,\ i\neq j\\
\{\alpha_i,\beta\} &= \alpha_i\beta+\beta\alpha_i=0
\end{aligned}$$

The coefficients $\vec{\alpha}$ and β that satisfy the above may be considered the matrix of forth-order. In our case, the Dirac expression described below is considered to be convenient:

$$\begin{aligned}
\alpha_i &= \begin{pmatrix} \boldsymbol{O}_2 & \boldsymbol{\sigma_i} \\ \boldsymbol{\sigma_i} & \boldsymbol{O}_2 \end{pmatrix} \equiv \rho_1\otimes\sigma_i\\
\beta &= \begin{pmatrix} \boldsymbol{I}_2 & \boldsymbol{O}_2 \\ \boldsymbol{O}_2 & -\boldsymbol{I}_2 \end{pmatrix} \equiv \rho_3\otimes\boldsymbol{I}_2
\end{aligned}$$

where $\vec{\sigma}$ and $\vec{\rho}$ are the Pauli matrices

$$\sigma_x=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_z=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They satisfy the following relation [109]

$$
\begin{aligned}
\sigma_i\sigma_j &= i\epsilon_{ijk}\sigma_k, \quad (i \neq j) \\
\sigma_i^2 &= \boldsymbol{I}_2 \\
[\sigma_i, \sigma_j] &= 2i\epsilon_{ijk}\sigma_k \\
\mathrm{Tr}\,\sigma_x &= \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_z = 0 \\
\det\sigma_x &= \det\sigma_y = \det\sigma_z = -1 \\
(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) &= (\vec{A}\cdot\vec{B})\boldsymbol{I}_2 + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})
\end{aligned}
$$

Here we use the sign $\otimes$ when we describe 4×4 matrices from a set of 2×2 matrix. (Tensor Product):

$$
\begin{aligned}
(A\otimes B)_{ia,jb} \equiv& A_{ij}B_{ab} \\
i,j =& 1,2 \quad a,b = 1,2 \\
(i,a),(j,b) =& (1,1),(1,2),(2,1),(2,2)
\end{aligned}
$$

Recall the multiplication of the block matrices, we may write

$$
(A\otimes B)(C\otimes D) = (AC\otimes BD)
$$

In another way, we may also understand from the equation

$$
\begin{aligned}
\{(A\otimes B)(C\otimes D)\}_{ia,jb} =& (A\otimes B)_{ia,kc}(C\otimes D)_{kc,jb} \\
=& A_{ik}B_{ac}C_{kj}D_{cb} = (AC)_{ij}(BD)_{ab} \\
=& (AC\otimes BD)_{ia,jb}
\end{aligned}
$$

Furthermore, [110]

109

$$
\begin{aligned}
(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) =& \sigma_i A_i \sigma_j B_j = \frac{1}{2}\{\sigma_i A_i\sigma_j B_j + \sigma_j A_j \sigma_i B_i\} \\
=& \frac{1}{2}\{\sum_{i=j}(\sigma_i\sigma_j A_i B_j + \sigma_j\sigma_i A_j B_i)\} + \sum_{i\neq j}(\sigma_i\sigma_j A_i B_j + \sigma_j\sigma_i A_j B_i)\} \\
=& \frac{1}{2}\left\{\sum_i 2\sigma_i^2 A_i B_i + \sum_{i\neq j}\sigma_i\sigma_j(A_iB_j - A_jB_i)\right\} \\
=& \vec{A}\cdot\vec{B} + i\frac{1}{2}\epsilon_{ijk}\sigma_k(A_iB_j - A_jB_i) \\
=& \vec{A}\cdot\vec{B} + i\epsilon_{ijk}\sigma_k A_i B_j \\
=& \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot\vec{A}\times\vec{B}
\end{aligned}
$$

110

$$
\mathrm{Tr}\,A\otimes B = \sum_{ia}(A\otimes B)_{ia,ia} = \sum_{ia} A_{ii}B_{aa} = \mathrm{Tr}\,A\,\mathrm{Tr}\,B
$$

$$\begin{aligned} \mathrm{Tr}\, A \otimes B =& \mathrm{Tr}\, A\, \mathrm{Tr}\, B \\ [A \otimes I, B \otimes C] =& AB \otimes C - BA \otimes C = [A, B] \otimes C \end{aligned}$$

The quantization $\vec{p} \to -i\hbar\vec{\nabla}$ via $H_{D,cl}$ is what we call the Dirac Hamiltonian H_D such that the Schroedinger equation is called the Dirac equation and written

$$\begin{aligned} H_D &= c\vec{\alpha} \cdot (\frac{\hbar}{i}\vec{\nabla} - e\vec{A}) + \beta mc^2 + e\phi \\ i\hbar\frac{\partial}{\partial t}\Psi(\vec{r}, t) &= H_D\Psi(\vec{r}, t) \end{aligned}$$

Here we bring the Dirac matrix $\gamma_\mu,\ \mu = 0, 1, 2, 3$ into the Dirac equation and rewrite which in [111]

$$\begin{aligned} \gamma^\mu &= (\gamma^0, \vec{\gamma}) \\ \gamma^0 &= \beta \\ \vec{\gamma} &= (\gamma_x, \gamma_y, \gamma_z) = \beta\vec{\alpha} = (\gamma^1, \gamma^2, \gamma^3) \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \end{aligned}$$

Note that the Hermitian for $\vec{\alpha}$ and β can be written

$$\begin{aligned} \gamma^{0\dagger} &= \gamma^0 \\ \gamma^{i\dagger} &= -\gamma^i \end{aligned}$$

We may simplify this in the form

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

[111]For example,

$$\gamma_1\gamma_1 = \beta\alpha_x\beta\alpha_x = -\beta\beta\alpha_x\alpha_x = -I$$

Given that we can write the Dirac equation [112]

$$\left\{ i\hbar\gamma^\mu(\partial_\mu + i\frac{e}{\hbar}A_\mu) - mc \right\}\Psi = 0$$
$$(i\hbar\gamma^\mu D_\mu - mc)\Psi = 0$$
$$D_\mu = \partial_\mu + i\frac{e}{\hbar}A_\mu$$

Note that there are four components in the wave function. For $i\hbar\frac{\partial}{\partial t}\Psi = H_D\Psi$, we can write

$$H_D = \gamma^0(-i\hbar c\vec{\gamma}\cdot\vec{\nabla} + mc^2)$$
$$i\hbar c\partial_0\Psi = H_D\Psi$$

5.2 Symmetry of Dirac Equation

The Conservation of Current

Consider now the Dirac equation and whose Hermitian conjugate, which gives [113]

$$\rho = \Psi^\dagger\Psi$$
$$\vec{j} = c\Psi^\dagger\vec{\alpha}\Psi$$

[112]

$$\left\{ i\hbar\frac{\partial}{\partial t} - c\boldsymbol{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla} - e\boldsymbol{A}) - \beta mc^2 - e\phi \right\}\Psi = 0$$
$$\frac{\gamma^0}{c}\times$$
$$\left\{ \gamma^0(i\hbar\frac{1}{c}\frac{\partial}{\partial t} - e\frac{1}{c}\phi) - \vec{\gamma}\cdot(-i\hbar\vec{\nabla} - e\vec{A}) - mc \right\}\Psi = 0$$
$$\left\{ i\hbar\gamma^0(\frac{\partial}{\partial(ct)} + ie\frac{1}{c\hbar}\phi) + i\hbar\vec{\gamma}\cdot(\vec{\nabla} - i\frac{e}{\hbar}\vec{A}) - mc \right\}\Psi = 0$$
$$\left\{ i\hbar\gamma^\mu(\partial_\mu + i\frac{e}{\hbar}A_\mu) - mc \right\}\Psi = 0$$
$$(i\hbar\gamma^\mu D_\mu - mc)\Psi = 0$$
$$D_\mu = \partial_\mu + i\frac{e}{\hbar}A_\mu$$

[113] The Dirac equation

$$i\hbar\frac{\partial\Psi}{\partial t} = c(-i\hbar\partial_i - eA_i)\alpha_i\Psi + (\beta mc^2 + e\phi)\Psi$$

whose Hermitian conjugate gives

$$-i\hbar\frac{\partial\Psi^\dagger}{\partial t} = c(i\hbar\partial_i - eA_i)\Psi^\dagger\alpha_i + \Psi^\dagger(\beta mc^2 + e\phi)$$

Thus, the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0$$

can be written.

In the covariant form we have $\bar{\Psi} = \Psi^{\dagger}\gamma^0$; the following relation for the conservation of current can be obtained: [114][115]

$$\begin{aligned} \partial_\mu j^\mu &= 0 \\ j^\mu &= \bar{\Psi}\gamma^\mu \Psi \end{aligned}$$

Thus,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}(\Psi^\dagger\Psi) &= i\hbar(\dot{\Psi}^\dagger\Psi + \Psi^\dagger\dot{\Psi}) \\ &= -ic\hbar\Big\{(\partial_i\Psi^\dagger\alpha_i)\Psi + \Psi^\dagger\alpha_i(\partial_i\Psi)\Big\} \\ &= -ic\hbar\partial_i(\Psi^\dagger\alpha_i\Psi) \end{aligned}$$

and

$$\begin{aligned} \rho &= \Psi^\dagger\Psi \\ \vec{j} &= c\Psi^\dagger\vec{\alpha}\Psi \end{aligned}$$

Hence

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0$$

[114]Given

$$i\hbar\gamma^\mu(\partial_\mu\Psi) - e\gamma^\mu A_\mu\Psi - mc\Psi = 0$$

the Hermitian conjugate may yield

$$-i\hbar(\partial_\mu\Psi^\dagger)\gamma^{\mu\dagger} - e\Psi^\dagger\gamma^{\mu\dagger}A_\mu - mc\Psi^\dagger = 0$$

Let us have $\bar{\Psi} = \Psi^\dagger\gamma^0$ and write

$$-i\hbar(\partial_\mu\bar{\Psi})\gamma^\mu - e\bar{\Psi}\gamma^\mu A_\mu - mc\bar{\Psi} = 0$$

Therefore the following relation of the conservation of current can be given

$$\begin{aligned} \partial_\mu j^\mu &= 0 \\ j^\mu &= \bar{\Psi}\gamma^\mu\Psi \end{aligned}$$

[115]In order to show the Lorentz invariance we must first show that the current j^μ is the invariant vector. Vice versa, we can say that the Lorentz invariance is being retained by experimentally identifying this conservation.

Conservation of Total Angular Momentum for Free Particles

Here we consider the free particles $\vec{A} = \vec{0}, \phi = 0$ in Dirac representation [116]

$$H = c\vec{\alpha} \cdot \vec{p} + \beta mc^2 = c\rho_1 \otimes \sigma_i p_i + \rho_3 mc^2$$

where we have

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ L_i &= \epsilon_{ijk} r_j p_k \end{aligned}$$

we may write

$$[\frac{\hbar}{2}\vec{\sigma} + \vec{L}, H] = 0$$

Thus,

$$\begin{aligned} [H, \vec{J}] &= 0 \\ \vec{J} &= \vec{L} + \vec{S} \\ \vec{S} &= \frac{\hbar}{2}\vec{\sigma} \end{aligned}$$

[116] For

$$H = c\vec{\alpha} \cdot \vec{p} + \beta mc^2 = c\rho_1 \otimes \sigma_i p_i + \rho_3 mc^2$$

in which

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ L_i &= \epsilon_{ijk} r_j p_k \end{aligned}$$

we may write

$$\begin{aligned} [L_i, H] &= c\rho_1 \otimes \sigma_\ell [\epsilon_{ijk} r_j p_k, p_\ell] = i\hbar c\rho_1 \otimes \sigma_\ell \epsilon_{ijk} \delta_{j\ell} p_k \\ &= i\hbar c\rho_1 \otimes \epsilon_{ijk} \sigma_j p_k = i\hbar c\rho_1 \otimes (\vec{\sigma} \times \vec{p})_i \end{aligned}$$

$$\begin{aligned} [AB, C] &= ABC - CAB \\ A[B, C] + [A, C]B &= ABC - ACB + (ACB - CAB) \end{aligned}$$

while we write

$$\begin{aligned} [\sigma_i, H] &= c\rho_1 \otimes [\sigma_i, \sigma_\ell] p_\ell \\ &= 2ic\rho_1 \otimes \epsilon_{i\ell k} \sigma_k p_\ell \\ &= -2ic\rho_1 \otimes (\vec{\sigma} \times \vec{p})_i \end{aligned}$$

Thus,

$$[\frac{\hbar}{2}\vec{\sigma} + \vec{L}, H] = 0$$

where we call $\vec{S}$ the spin, and therefore the total angular momentum $\vec{J}$ becomes the conserved quantity.

Conservation of Energy and Momentum for Free Particles

For the free particles $A^\mu = 0$, we can write

$$\begin{aligned} H_D =& c\rho_1 \otimes \sigma_i p_i + \rho_3 mc^2 \\ [H_D, H_D] =& 0 \\ [H_D, \vec{p}] =& \vec{0} \end{aligned}$$

5.2.1 The Lorentz Invariance

The Lorentz transformation

$$\begin{aligned} x'^\mu &= \Omega^\mu{}_\nu x^\nu \\ x'^\mu \Omega_\mu{}^\kappa &= x^\kappa \end{aligned}$$

gives D_μ, which is transforming as the covariance vector thus, [117] ($D_\mu = D'_\nu \Omega^\nu{}_\mu$)

$$\hat{\gamma}^\mu = \Omega^\mu{}_\nu \gamma^\nu$$

giving [118]

$$(i\hbar\hat{\gamma}^\nu D'_\nu - mc)\Psi(x) = 0$$

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$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu = \Omega_\mu{}^\nu \partial_\nu$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Omega^\nu{}_\mu \partial'_\nu = \partial'_\nu \Omega^\nu{}_\mu$$

while the covariance of A_μ gives

$$\begin{aligned} A'_\mu(x') &= \Omega_\mu{}^\nu A_\nu(x) \\ A'_\mu(x')\Omega^\mu{}_\kappa &= \Omega_\mu{}^\nu A_\nu(x)\Omega^\mu{}_\kappa = g_{\mu\rho}\Omega^{\rho\nu} A_\nu(x) g^{\mu\tau}\Omega_{\tau\kappa} = \delta_\rho{}^\tau \Omega^{\rho\nu} A_\nu(x)\Omega_{\tau\kappa} \\ &= \Omega^{\rho\nu} A_\nu(x)\Omega_{\rho\kappa} = \delta^\nu{}_\kappa A_\nu(x) = A_\kappa(x) \end{aligned}$$

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$$\begin{aligned} 0 &= (i\hbar\gamma^\mu D_\mu - mc)\Psi(x) = (i\hbar\gamma^\mu D'_\nu \Omega^\nu{}_\mu - mc)\Psi(x) \\ &= \big(i\hbar(\Omega^\nu{}_\mu \gamma^\mu) D'_\nu \Omega^\nu{}_\mu - mc\big)\Psi(x) \\ &= (i\hbar\hat{\gamma}^\nu D'_\nu - mc)\Psi(x) \end{aligned}$$

Now let us have [119]

$$\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2g^{\mu\nu}$$

which indicates the existence of the regular matrix Λ, and for all μ we know that there is Λ that satisfies

$$\hat{\gamma}^\mu = \Lambda^{-1}\gamma^\mu\Lambda$$

For the in depth explanation, which is covered in our later discussion, and given the fact, the Dirac equation forms the Lorentz covariance as described in the following: [120]

$$(i\hbar\gamma^\mu D'_\mu - mc)\Psi'(x') = 0$$

Thus,

$$\Psi'(x') = \Lambda\Psi(x)$$

so, we can write

$$x' = \mathcal{L}x, \quad x'^\mu = \Omega^\mu{}_\nu x^\nu$$

Therefore,

$$\Psi'(x') = (\mathcal{L}\Psi)(x') = (\mathcal{L}\Psi)(\mathcal{L}x) = \Lambda\Psi(x)$$

A specific structure of the transformation matrix Here we elaborate on Λ used in our discussion for a specific construction. First, consider the infinitesimal Lorentz transformation

$$\Omega^\mu{}_\nu = g^\mu{}_\nu + \delta\Omega^\mu{}_\nu$$

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$$\begin{aligned}\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} &= \Omega^\mu{}_\kappa\Omega^\nu{}_\rho\{\gamma^\kappa, \gamma^\rho\} = 2\Omega^\mu{}_\kappa\Omega^\nu{}_\rho g^{\kappa\rho} \\ &= 2\Omega^\mu{}_\kappa\Omega^{\nu\kappa} = 2g^{\mu\tau}\Omega_{\tau\kappa}\Omega^{\nu\kappa} = 2g^{\mu\tau}\delta_\tau{}^\nu = 2g^{\mu\nu}\end{aligned}$$

120

$$\begin{aligned}&(i\hbar\Lambda^{-1}\gamma^\mu\Lambda D'_\mu - mc)\Psi = 0 \\ &\quad (i\hbar\gamma^\mu\Lambda D'_\mu - mc\Lambda)\Psi \equiv (i\hbar\gamma^\mu D'_\mu - mc)\Psi'(x') = 0\end{aligned}$$

Given that we have up to the degree of first-order ${\Omega^{\mu}}_{\nu}{\Omega_{\mu}}^{\lambda} = g_{\nu}^{\lambda}$ for the infinitesimal quantity, [121] we can write

$$\delta\Omega_{\lambda\nu} = -\,\delta\Omega_{\nu\lambda}$$

Now, let us rewrite $\Lambda^{-1}\gamma^{\mu}\Lambda = {\Omega^{\mu}}_{\nu}\gamma^{\nu}$. To do so, we begin by writing down

$$\begin{aligned}
{\Omega^{\mu}}_{\nu} =& {g^{\mu}}_{\nu} + \delta{\Omega^{\mu}}_{\nu} \\
{\Omega^{\mu}}_{\nu}\gamma^{\nu} =& \gamma^{\mu} + \delta{\Omega^{\mu}}_{\nu}\gamma^{\nu} \\
\Lambda =& I + \delta\Lambda \quad Towhich, wemaywrite \\
(I - \delta\Lambda)\gamma^{\mu}(I + \delta\Lambda) =& \gamma^{\mu} - [\delta\Lambda, \gamma^{\mu}]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta{\Omega^{\mu}}_{\nu}\gamma^{\nu} =& -\,[\delta\Lambda, \gamma^{\mu}] \\
\delta\Omega_{\mu\nu}\gamma^{\nu} =& -\,[\delta\Lambda, \gamma_{\mu}]
\end{aligned}$$

and

$$\delta\Lambda = -\,\frac{i}{4}\sigma^{\kappa\nu}\delta\Omega_{\kappa\nu}$$

Given the antisymmetric property of $\delta\Omega_{\mu\nu}$ we suppose

$$\sigma^{\mu\nu} = -\,\sigma^{\nu\mu}$$

without losing the generality, and being aware of the antisymmetric property, we can write

$$\begin{aligned}
\delta\Omega_{\mu\nu}\gamma^{\nu} =& \frac{i}{4}[\sigma^{\kappa\nu}, \gamma_{\mu}]\delta\Omega_{\kappa\nu} \\
[\sigma^{\kappa\nu}, \gamma_{\mu}] =& -\,2i(g_{\mu}^{\kappa}\gamma^{\nu} - g_{\mu}^{\nu}\gamma^{\kappa})
\end{aligned}$$

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$$\begin{aligned}
g_{\nu}^{\lambda} =& ({g^{\mu}}_{\nu} + \delta{\Omega^{\mu}}_{\nu})({g_{\mu}}^{\lambda} + \delta{\Omega_{\mu}}^{\lambda}) \\
=& g_{\nu}^{\lambda} + \delta{\Omega^{\lambda}}_{\nu} + \delta{\Omega_{\nu}}^{\lambda} \\
0 =& \delta{\Omega^{\lambda}}_{\nu} + \delta{\Omega_{\nu}}^{\lambda} \\
0 =& \delta\Omega_{\lambda\nu} + \delta\Omega_{\nu\lambda}
\end{aligned}$$

We can show that the following relation for $\sigma^{\mu\nu}$ being satisfied: [122]

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

By integrating the above we obtain [123]

$$\begin{aligned}\boldsymbol{\Omega} =& e^{\boldsymbol{\omega}},\\ \tilde{\boldsymbol{\omega}} =& -\boldsymbol{\omega}\end{aligned}$$

For the above equations we may write down ($\boldsymbol{\omega}$: real antisymmetric) [124]

---

[122]

$$\begin{aligned}\sigma^{\mu\nu} =& \frac{i}{2}[\gamma^\mu, \gamma^\nu]\\ \because [\gamma^\kappa\gamma^\nu, \gamma^\mu] =& \gamma^\kappa\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\kappa\gamma^\nu\\ =& \gamma^\kappa(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu}) - \gamma^\mu\gamma^\kappa\gamma^\nu\\ =& -\gamma^\kappa\gamma^\mu\gamma^\nu + 2\gamma^\kappa g^{\mu\nu} - \gamma^\mu\gamma^\kappa\gamma^\nu\\ =& -2g^{\kappa\mu}\gamma^\nu + 2\gamma^\kappa g^{\mu\nu}\\ [[\gamma^\kappa, \gamma^\nu], \gamma^\mu] =& -2g^{\kappa\mu}\gamma^\nu + 2\gamma^\kappa g^{\mu\nu} - (-2g^{\nu\mu}\gamma^\kappa + 2\gamma^\nu g^{\mu\kappa})\\ =& -4g^{\kappa\mu}\gamma^\nu + 4\gamma^\kappa g^{\mu\nu}\\ [[\gamma^\kappa, \gamma^\nu], \gamma_\mu] =& -4g^\kappa_\mu\gamma^\nu + 4\gamma^\kappa g^\nu_\mu\\ [\frac{i}{2}[\gamma^\kappa, \gamma^\nu], \gamma_\mu] =& -2i(g^\kappa_\mu\gamma^\nu - \gamma^\kappa g^\nu_\mu)\end{aligned}$$

[123]

$$\begin{aligned}\boldsymbol{\Omega} =& e^{\boldsymbol{\omega}}\\ \boldsymbol{\Omega}\tilde{\boldsymbol{\Omega}} =& \boldsymbol{I} gives\\ \tilde{\boldsymbol{\omega}} =& -\boldsymbol{\omega}\end{aligned}$$

To express the components, given

$$(\tilde{\Omega})^\mu{}_\nu = \Omega_\nu{}^\mu$$

which yields

$$\begin{aligned}(\Omega\tilde{\Omega})^\mu{}_\nu =& \Omega^\mu{}_\kappa\Omega_\nu{}^\kappa = \delta^\mu{}_\nu\\ (\Omega^{-1})^\kappa{}_\nu =& \Omega_\nu{}^\kappa\\ (\tilde{\omega})^\mu{}_\nu =& \omega_\nu{}^\mu = -\omega^\mu{}_\nu\\ (e^\omega)^\mu{}_\kappa(e^\omega)_\nu{}^\kappa =& (e^\omega)^\mu{}_\kappa(\tilde{e^\omega})^\kappa{}_\nu = (e^\omega)^\mu{}_\kappa(e^{\tilde{\omega}})^\kappa{}_\nu = \delta^\mu{}_\nu\end{aligned}$$

[124] First we begin with writing down

$$\hat{\gamma}^\kappa = (e^{t\omega})^\kappa{}_\lambda\gamma^\lambda\Big|_{t=1}$$

$$\Lambda = e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}$$

While [125]

$$\Lambda^\dagger = \gamma^0 \Lambda^{-1} \gamma^0$$

---

and

$$\Lambda(t) = e^{-\frac{it}{4}\sigma^{\mu\nu}\omega_{\mu\nu}} = e^{-\frac{it}{4}\sigma_\mu{}^\nu \omega^\mu{}_\nu}$$

to give

$$\begin{aligned}
\Gamma^\kappa(t) =& \Lambda(t)^{-1}\gamma^\kappa\Lambda(t) = e^{+\frac{it}{4}\sigma_\mu{}^\nu\omega^\mu{}_\nu}\gamma^\kappa e^{-\frac{it}{4}\sigma_\mu{}^\nu\omega^\mu{}_\nu} \\
\frac{\partial\Gamma^\kappa}{\partial t} =& \frac{i}{4}\Lambda^{-1}[\sigma_\mu{}^\nu, \gamma^\kappa]\Lambda\omega^\mu{}_\nu = \frac{1}{2}\Lambda^{-1}(g^\kappa_\mu\gamma^\nu - g^{\kappa\nu}\gamma_\mu)\Lambda\omega^\mu{}_\nu \\
=& \frac{1}{2}(g^\kappa_\mu\Gamma^\nu - g^{\kappa\nu}\Gamma_\mu)\omega^\mu{}_\nu = \frac{1}{2}(\Gamma^\nu\omega^\kappa{}_\nu - \Gamma_\mu\omega^{\mu\kappa}) = \frac{1}{2}(\Gamma^\nu\omega^\kappa{}_\nu - \Gamma^\mu\omega_\mu{}^\kappa) \\
=& \frac{1}{2}(\Gamma^\nu\omega^\kappa{}_\nu + \Gamma^\mu\omega^\kappa{}_\mu) = \omega^\kappa{}_\mu\Gamma^\mu
\end{aligned}$$

Where $t = 0$, note for $\Gamma^\mu(0) = \gamma^\mu$, the solution of the simultaneous differential equation is given

$$\Gamma^\mu = (e^{t\omega})^\mu{}_\nu\gamma^\nu$$

While $t = 1$, the solution is given

$$\Gamma^\mu(1) = \hat{\gamma}^\mu$$

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$$\begin{aligned}
\sigma^{\mu\nu\dagger} =& \left(\frac{i}{2}[\gamma^\mu, \gamma^\nu]\right)^\dagger = -\frac{i}{2}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}] \\
=& \frac{i}{2}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}] = \gamma^0\frac{i}{2}[\gamma^\mu, \gamma^\nu]\gamma^0 \\
=& \gamma^0\sigma^{\mu\nu}\gamma^0 \\
\Lambda^\dagger =& e^{\frac{i}{4}(\sigma^{\mu\nu})^\dagger\omega_{\mu\nu}} \\
=& \gamma^0 e^{\frac{i}{4}(\sigma^{\mu\nu})\omega_{\mu\nu}}\gamma^0 \\
=& \gamma^0\Lambda^{-1}\gamma^0
\end{aligned}$$

For this Lorentz transformation, we can write the current

$$
\begin{aligned}
\Psi =&\Lambda^{-1}\Psi' = \gamma^0\Lambda^\dagger\gamma^0\Psi' \\
j^\mu =&\bar{\Psi}\gamma^\mu\Psi \\
=&\Psi^\dagger\gamma^0\gamma^\mu\Psi \\
=&\Psi'^\dagger\gamma^0\Lambda\gamma^0(\gamma^0\gamma^\mu)\Lambda^{-1}\Psi \\
=&\Psi'^\dagger\gamma^0\Lambda\gamma^\mu\Lambda^{-1}\Psi' \\
=&\bar{\Psi}'\Lambda\gamma^\mu\Lambda^{-1}\Psi'
\end{aligned}
$$

Recall our discussion in the previous section, we can express

$$
\begin{aligned}
\Omega^\mu{}_\nu\gamma^\nu =&\Lambda^{-1}\gamma^\mu\Lambda \\
\Omega^\mu{}_\nu\Lambda\gamma^\nu\Lambda^{-1} =&\gamma^\mu \\
\Omega_\mu{}^\kappa\Omega^\mu{}_\nu\Lambda\gamma^\nu\Lambda^{-1} =&g^\kappa_\nu\Lambda\gamma^\nu\Lambda^{-1} = \Lambda\gamma^\kappa\Lambda^{-1} = \\
=&\Omega_\mu{}^\kappa\gamma^\mu = \gamma^\mu\Omega_\mu{}^\kappa
\end{aligned}
$$

Thus, [126]

$$
\begin{aligned}
j'^\kappa =&\Omega^\kappa{}_\mu j^\mu \\
j'^\mu =&\bar{\Psi}'\gamma^\mu\Psi'
\end{aligned}
$$

This implies that the current is capable of transforming itself into the invariant vector such that the conservation $\partial_\mu j^\mu = 0$ can be regarded as the Lorentz invariant.

5.3 The Plane-wave Solutions for the Free Dirac Equation

In this section, we consider the solutions for Dirac equation where $A^\mu = 0$. Let us write the Dirac Hamiltonian

$$
H =c\vec{\alpha}\cdot\frac{\vec{\nabla}}{i} + \beta mc^2 = c\rho_1\otimes\vec{\sigma}\cdot\vec{p} + \rho_3 mc^2
$$

such that the Dirac equation can be written

$$
i\hbar c\partial_0\Psi =H\Psi
$$

[126]

$$
\begin{aligned}
j^\mu =&j'^\nu\Omega_\nu{}^\mu \\
\Omega^\kappa{}_\mu j^\mu =&\Omega^\kappa{}_\mu\Omega_\nu{}^\mu j'^\nu = g^\kappa_\nu j'^\nu = j'^\kappa
\end{aligned}
$$

Rewrite the above and give

$$\begin{aligned}
\Psi^{(+)}(x) =& e^{-ik_\mu x^\mu} u(k) \\
\Psi^{(-)}(x) =& e^{+ik_\mu x^\mu} v(k) \\
-k^\mu x^\mu =& -k^0 x^0 + k^i x^i = \vec{k}\cdot\vec{r} - \omega t \\
(k_x, k_y, k_z) =& (k^1, k^2, k^3) = (-k_1, -k_2, -k_3) \\
k_0 =& k^0 = \frac{\omega}{c}
\end{aligned}$$

Thus,

$$H^2 = (c^2\vec{p}^2 + m^2c^4)\mathbf{1}_4$$

Note the above, and obtain the following relation for the solutions of the plane waves:[127]

$$\begin{aligned}
\vec{p}\Psi^{(\pm)} =& \pm\hbar\vec{k}\Psi^{(\pm)} \\
H\Psi^{(\pm)} =& \pm E\Psi^{(\pm)} \\
Hu =& +Eu \\
Hv =& -Ev \\
E =& c\hbar k_0 = c\hbar k^0 = \hbar\omega \\
\hbar k_0 =& \sqrt{\hbar\vec{k}^{\,2} + m^2c^2} \\
k_\mu k^\mu =& \left(\frac{mc}{\hbar}\right)^2
\end{aligned}$$

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$$\begin{aligned}
H^2 =& c^2\rho^2 \otimes (\vec{\sigma}\cdot\vec{p})^2 + \rho_3^2 m^2c^4 + 2mc^2(\rho_1\rho_3 + \rho_3\rho_1)\vec{\sigma}\cdot\vec{p} \\
=& (c^2\vec{p}^2 + m^2c^4)\mathbf{1}_4
\end{aligned}$$

or

$$\begin{aligned}
H =& \gamma^0(-i\hbar c\vec{\gamma}\cdot\vec{\nabla} + mc^2) \\
H^2 =& \gamma^0(-i\hbar c\vec{\gamma}\cdot\vec{\nabla} + mc^2)\gamma^0(-i\hbar c\vec{\gamma}\cdot\vec{\nabla} + mc) \\
=& -\hbar^2c^2\gamma^0\gamma^i\gamma^0\gamma^j(\vec{\nabla})_i(\vec{\nabla})_j + m^2c^4 - i\hbar mc^2(\gamma^i\gamma^0 + \gamma^i\gamma^0)(\vec{\nabla})_i \\
=& -\hbar^2c^2(-\gamma^i)\gamma^j(\vec{\nabla})_i(\vec{\nabla})_j + m^2c^4 \\
=& -\hbar^2c^2\vec{\nabla}^2 + m^2c^4 = c^2\vec{p}^2 + m^2c^4
\end{aligned}$$

$$\vec{p}\Psi^{(\pm)} = \frac{\hbar}{i}\vec{\nabla}\Psi^{(\pm)} = \frac{\hbar}{i}(\mp i)(k_1, k_2, k_3)\Psi^{(\pm)} = \pm\hbar(k^1, k^2, k^3)\Psi^{(\pm)} = \pm\hbar\vec{k}\Psi^{(\pm)}$$

While we let $(i\hbar\gamma^\mu\partial_m u - mc)\Psi^{(\pm)} = 0$ given by the Dirac equation $\not{k} = k_\mu\gamma^\mu$ such that [128]

$$
\begin{aligned}
(\hbar\not{k} - mc)u =& 0 \\
(\hbar\not{k} + mc)v =& 0
\end{aligned}
$$

5.3.1 In the Case of $m \neq 0$

If we take the inertial system $\vec{v} = 0$, $k^\mu = (\frac{mc}{\hbar}, 0, 0, 0)$ which stays stationary, the complete system $u^\alpha_{\text{rest}}, v^\alpha_{\text{rest}}, \alpha = 1, 2$ can be given [129]

$$
\begin{aligned}
u^1_{\text{rest}} =& \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^2_{\text{rest}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^1_{\text{rest}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^2_{\text{rest}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
\bar{u}^\alpha u^\beta =& \delta_{\alpha\beta}, \quad \bar{v}^\alpha v^\beta = -\delta_{\alpha\beta}, \quad \bar{u}^\alpha v^\beta = \bar{v}^\alpha u^\beta = 0 \\
u^\alpha_{\text{rest}} =& \begin{pmatrix} \psi^\alpha_{\text{rest}} \\ 0 \end{pmatrix}, \quad v^\alpha_{\text{rest}} = \begin{pmatrix} 0 \\ \chi^\alpha_{\text{rest}} \end{pmatrix}
\end{aligned}
$$

From which we determine the general solutions for the plane waves via Lorentz

---

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$$
\begin{aligned}
(\pm\hbar\not{k} - mc)\Psi^{(\pm)}(x) =& 0, \quad \not{k} = k_\mu\gamma^\mu \\
(\hbar\not{k} - mc)u =& 0 \\
(\hbar\not{k} + mc)v =& 0
\end{aligned}
$$

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$$
\begin{aligned}
mc(\gamma^0 - 1)u_{\text{rest}} =& mc \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & -2 & \\ & & & -2 \end{pmatrix} u_{\text{rest}} = 0 \\
mc(\gamma^0 + 1)v_{\text{rest}} =& mc \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} v_{\text{rest}} = 0
\end{aligned}
$$

transformation. We begin by the equations [130]

$$
\begin{aligned}
\not{a}\not{b} =& - i a_\mu b_\nu \sigma^{\mu\nu} + a_\mu b^\nu \\
\not{k}\not{k} =& k_\mu k^\mu = k^2
\end{aligned}
$$

which gives

$$
\begin{aligned}
u^\alpha =& \frac{1}{mc}(\hbar\not{k} + mc) u^\alpha_{\text{rest}} \\
=& \frac{1}{mc}\begin{pmatrix} (\hbar k_0 + mc)\psi^\alpha_{\text{rest}} \\ \gamma_i \hbar k^i \psi^\alpha_{\text{rest}} \end{pmatrix} = \frac{1}{mc}\begin{pmatrix} (\frac{E}{c} + mc)\psi^\alpha_{\text{rest}} \\ (\vec{\sigma}\cdot\vec{p})\psi^\alpha_{\text{rest}} \end{pmatrix}
\end{aligned}
$$

5.4 The Non-relativistic Limit

The four components spinor can be written by the two components spinor ψ and χ:

$$
\Psi(x) = \begin{pmatrix} \psi(x) \\ \chi(x) \end{pmatrix}
$$

Let us write the Dirac equation in the forms

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \psi \\ \chi \end{pmatrix} &= \begin{pmatrix} mc^2 + e\phi & cP \\ cP & -mc^2 + e\phi \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} \\
P &= \vec{\alpha}\cdot\vec{\pi} = \vec{\sigma}\cdot(\vec{p} - e\vec{A})
\end{aligned}
$$

In steady states, we obtain

$$
\begin{aligned}
\psi(x) &= e^{-iEt/\hbar}\psi(\vec{r}) \\
\chi(x) &= e^{-iEt/\hbar}\chi(\vec{r})
\end{aligned}
$$

yielding

$$
\begin{aligned}
(mc^2 + e\phi)\psi + cP\chi &= E\psi \\
cP\psi + (-mc^2 + e\phi)\chi &= E\chi
\end{aligned}
$$

[130]

$$
\begin{aligned}
\not{a}\not{b} =& a_\mu \gamma^\mu b_\nu \gamma^\nu = \frac{1}{2}(a_\mu b_\nu \gamma^\mu\gamma^\nu + a_\nu b_\mu \gamma^\nu\gamma^\mu) = \frac{1}{2}\big(a_\mu b_\nu \gamma^\mu\gamma^\nu + a_\nu b_\mu(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu})\big) \\
=& \frac{1}{2}(a_\mu b_\nu - a_\nu b_\mu)\gamma^\mu\gamma^\nu + a_\mu b^\nu = \frac{1}{2}a_\mu b_\nu[\gamma^\mu,\gamma^\nu] + a_\mu b^\nu \\
=& - i a_\mu b_\nu \sigma^{\mu\nu} + a_\mu b^\nu \\
\sigma^{\mu\nu} =& \frac{i}{2}[\gamma^\mu,\gamma^\nu]
\end{aligned}
$$

To consider the non-relativistic limit

$$e\phi \quad << \quad mc^2, \quad \frac{P^2}{2m} << mc^2, \quad E \approx mc^2$$

we transform the Dirac equation into a much more convenient form

$$W \quad = \quad E - mc^2$$

Thus, given the second equation we can write down

$$\begin{aligned} \chi &= \frac{c}{2M'c^2} P\psi = \frac{1}{2M'c} P\psi \\ 2M'c^2 &= E + mc^2 - e\phi = 2mc^2 + W - e\phi \\ M' &= m + \frac{1}{2c^2}(W - e\phi) \end{aligned}$$

From these equations the Dirac equation can be accurately rewritten in the form [131]

$$(P\frac{1}{2M'}P + e\phi)\psi \quad = \quad W\psi$$

The Lowest Order Approximation

For the lowest order approximation we suppose

$$M' \quad = \quad m$$

This gives (Schroedinger approximation)

$$\begin{aligned} H_{sh}\psi &= W\psi \\ H_{sh} &= \frac{1}{2m}P^2 + e\phi \end{aligned}$$

Here note that [132]

$$P^2 \quad = \quad \vec{\pi}^2 - e\hbar\vec{\sigma}\cdot\vec{B}, \quad \vec{B} = \mathrm{rot}\, A$$

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$$\begin{aligned} (mc^2 + e\phi)\psi + P\frac{1}{2M'}P\psi &= E\psi \\ (P\frac{1}{2M'}P + e\phi)\psi &= W\psi \end{aligned}$$

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$$\begin{aligned} P^2 &= (\vec{\sigma}\cdot\vec{\pi})^2 = (\sigma_i\pi^i)(\sigma_j\pi^j) = \vec{\pi}^2 + \frac{1}{2}(\sigma_i\sigma_j - \sigma_j\sigma_i)\pi^i\pi^j \\ &= \vec{\pi}^2 + i\epsilon_{ijk}\sigma_i\pi_j\pi_k = \vec{\pi}^2 + i\epsilon_{ijk}\sigma_i(p_j - eA_j)(p_k - eA_k) \\ &= \vec{\pi}^2 - ie\epsilon_{ijk}\sigma_i(p_jA_k) = \vec{\pi}^2 - ie\epsilon_{ijk}\sigma_i\frac{\hbar}{i}(\partial_jA_k) \\ &= \vec{\pi}^2 - e\hbar\vec{\sigma}\cdot\vec{B}, \quad \vec{B} = \mathrm{rot}\, A \end{aligned}$$

Thus,

$$\begin{aligned}
H_{sh} &= \frac{(\vec{p}-e\vec{A})^2}{2m} + e\phi + \vec{\mu}\cdot\vec{B} \\
\vec{\mu} &= -\frac{e\hbar}{2m}\vec{\sigma} \\
&= -g\mu_B \vec{S}/\hbar, \quad (\vec{S} = \frac{\hbar}{2}\sigma) \\
Where Borh magneton is \ \mu_B &= \frac{e\hbar}{2m} \\
, and so call \ g \ factor is \ g &= 2
\end{aligned}$$

.

The Approximation to $\frac{v^2}{c^2}$

In our next step, we raise the order of approximation [133] to

$$\frac{1}{M'} \approx \frac{1}{m} - \frac{1}{2m^2c^2}(W - e\phi)$$

Here we make an estimate of $W - e\phi \approx mv^2$, where we take the value up to $\frac{v^2}{c^2}$ such that we can write

$$P\frac{1}{2M'}P = \frac{P^2}{2m} - \frac{1}{4m^2c^2}WP^2 + \frac{e}{4m^2c^2}P\phi P$$

and which gives

$$\left(\frac{P^2}{2m} + e\phi + \frac{e}{4m^2c^2}P\phi P\right)\psi = W(1 + \frac{1}{4m^2c^2}P^2)\psi$$

Now we consider the normalization condition such that

$$\chi = \frac{1}{2mc}P\psi$$

For this we can write

$$\begin{aligned}
1 &= \int d^3r\, \Psi^\dagger\Psi = \int d^3r\, (\psi^\dagger\psi + \chi^\dagger\chi) \\
&= \int d^3r\, \psi^\dagger(1 + \frac{1}{4m^2c^2}P^2)\psi
\end{aligned}$$

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$$\begin{aligned}
\frac{1}{M'} &= \frac{1}{m}(1 + \frac{1}{2mc^2}(W - e\phi))^{-1} \\
&= \frac{1}{m} - \frac{1}{2m^2c^2}(W - e\phi) + o(\frac{v^2}{c^2}) \\
&\approx \frac{1}{m} - \frac{1}{2m^2c^2}(W - e\phi)
\end{aligned}$$

Therefore if we let the normalized wavefunction ψ_N in two components to be

$$\begin{aligned}\psi_N &= \Omega\psi \\ 1 &= \int d^3r\, \psi_N^\dagger \psi_N\end{aligned}$$

then we may also have

$$\Omega = 1 + \frac{1}{8m^2c^2}P^2$$

The equation for ψ_N can be given [134] [135]

$$\left(\frac{P^2}{2m} + e\phi - \frac{P^4}{8m^3c^2} - \frac{e}{8m^2c^2}[P,[P,\phi]]\right)\psi_N = W\psi_N$$

When we look into the degree of order, the below indicates that there are the values up to $\frac{v^2}{c^2}$:

$$\begin{aligned}\frac{e}{8m^2c^2}[P,[P,\phi]] &\approx \frac{mv^2(mv)^2}{m^2c^2} = \frac{1}{2}mv^2\cdot\left(\frac{v^2}{c^2}\right) \\ \frac{1}{8m^3c^2}P^4 &\approx \frac{(mv)^4}{m^3c^2} = \frac{1}{2}mv^2\cdot\left(\frac{v^2}{c^2}\right)\end{aligned}$$

[134] Given $\{A^2, B\} - 2ABA = A^2B - BA^2 - 2ABA$, $[A,[A,B]] = A(AB-BA) - (AB-BA)A = A^2B - 2ABA + BA^2$ we may use $\{A^2,B\} - 2ABA = [A,[A,B]]$

[135]

$$\begin{aligned}\left(\frac{P^2}{2m} + e\phi + \frac{e}{4m^2c^2}P\phi P\right)\Omega^{-1}\psi_N &= W\Omega\psi_N \\ \Omega^{-1}\left(\frac{P^2}{2m} + e\phi + \frac{e}{4m^2c^2}P\phi P\right)\Omega^{-1}\psi_N &= W\psi_N \\ \left(\frac{P^2}{2m} + e\phi - \frac{P^4}{8m^3c^2} - \frac{e}{8m^2c^2}\{\phi,P^2\} + \frac{e}{4m^2c^2}P\phi P\right)\psi_N + o(\frac{v^2}{c^2}) &= W\psi_N \\ \left(\frac{P^2}{2m} + e\phi - \frac{P^4}{8m^3c^2} - \frac{e}{8m^2c^2}[P,[P,\phi]]\right)\psi_N &= W\psi_N\end{aligned}$$

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$$
\begin{aligned}
P^4 &= [\vec{\pi}^2 - e\hbar(\vec{\sigma}\cdot\vec{B})]^2 \\
[P,\phi] &= [\sigma_j(p_j - eA_j),\phi] = \sigma_j(p_j\phi) = -i\hbar\sigma_j\partial_j\phi
\end{aligned}
$$

In the stationary electric fields, which is given by $\vec{E} = -\vec{\nabla}\phi$, we can write

$$
[P,[P,\phi]] = \hbar^2 \mathrm{div}\,\vec{E} + 2\hbar\vec{\sigma}\cdot\vec{E}\times\vec{\pi}
$$

The approximation (Pauli approximation) can be made to the degree of order we obtained in the above so that we write

$$
\begin{aligned}
H_{pauli}\psi_N &= W\psi_N \\
H_{pauli} &= H_{sh} + H_c \\
H_{sh} &= \frac{1}{2m}(\vec{\pi}^2 - e\hbar\vec{\sigma}\cdot\vec{B})^2 + e\phi = \frac{1}{2m}\vec{\pi}^2 + e\phi - \frac{e\hbar}{2m}\vec{\sigma}\cdot\vec{B} \\
H_c &= -\frac{(\vec{\pi}^2 - e\hbar\sigma\cdot\vec{B})^2}{8m^3c^2} - \frac{e\hbar^2}{8m^2c^2}\mathrm{div}\,\vec{E} - \frac{e\hbar}{4m^2c^2}\vec{\sigma}\cdot\vec{E}\times\vec{\pi}
\end{aligned}
$$

Now we consider the non-relativistic limit for the classical Hamiltonian, we can write

$$
\begin{aligned}
H_{cl} &= c\sqrt{\vec{\pi}^2 + m^2c^2} + e\phi = mc^2\sqrt{1 + \frac{\vec{\pi}^2}{m^2c^2}} + e\phi \\
&\approx e\phi + mc^2\left(1 + \frac{1}{2}\frac{\pi^2}{m^2c^2} - \frac{1}{8}\frac{\pi^4}{m^4c^4}\right) \\
&= e\phi + mc^2 + \frac{\vec{\pi}^2}{2m} - \frac{\vec{\pi}^4}{2m^3c^4}
\end{aligned}
$$

where we define $\vec{\pi}^2 \to \vec{\pi}^2 - e\hbar\vec{\sigma}\cdot\vec{B}$, which includes the effects by the spin, the first term of H_c can be regarded as the correction term for the relativistic kinetic energy. The second term of the equation is called the Darwin term.

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$$
\begin{aligned}
P^4 &= [\vec{\pi}^2 - e\hbar(\vec{\sigma}\cdot\vec{B})]^2 \\
[P,\phi] &= [\sigma_j(p_j - eA_j),\phi] = \sigma_j(p_j\phi) = -i\hbar\sigma_j\partial_j\phi \\
[P,[P,\phi]] &= -i\hbar[\sigma_i(p_i - eA_i),\sigma_j\partial_j\phi] \\
&= -\hbar^2[\sigma_i\partial_i,\sigma_j\partial_j\phi] + ie\hbar[\sigma_iA_i,\sigma_j\partial_j\phi] \\
&= -\hbar^2\sigma_i\sigma_j\partial_i\partial_j\phi - \hbar^2\sigma_i\sigma_j(\partial_j\phi)\partial_i + \hbar^2\sigma_j\sigma_i(\partial_j\phi)\partial_i + ie\hbar[\sigma_i,\sigma_j]A_i\partial_j\phi \\
&= -\hbar^2\Delta\phi - \hbar^2[\sigma_i,\sigma_j](\partial_j\phi)\partial_i - 2e\hbar\epsilon_{ijk}\sigma_kA_i(\partial_j\phi) \\
&= -\hbar^2\Delta\phi - 2i\hbar^2\epsilon_{ijk}\sigma_k(\partial_j\phi)\partial_i - 2e\hbar\epsilon_{ijk}\sigma_kA_i(\partial_j\phi) \\
&= \hbar^2\mathrm{div}\,\vec{E} - 2i\hbar^2\vec{\sigma}\cdot\vec{E}\times\vec{\nabla} + 2e\hbar\sigma\cdot\vec{A}\times\vec{E} \\
&= \hbar^2\mathrm{div}\,\vec{E} + 2\hbar\vec{\sigma}\cdot\vec{E}\times(\vec{p} - e\vec{A}) \\
&= \hbar^2\mathrm{div}\,\vec{E} + 2\hbar\vec{\sigma}\cdot\vec{E}\times\vec{\pi}
\end{aligned}
$$

For the last term of the equation, when we consider the central force field,
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$$
\begin{aligned}
e\phi(\vec{r}) &= V(r), \quad \vec{A} = \vec{0} \\
H_{LS} &\equiv -\frac{e\hbar}{4m^2c^2}\vec{\sigma}\cdot\vec{E}\times\vec{\pi} = \frac{\hbar}{4m^2c^2}\frac{1}{r}\frac{\partial V}{\partial r}\vec{\sigma}\cdot(\vec{r}\times\vec{p}) \\
&= \left(\frac{1}{2m^2c^2}\frac{1}{r}\frac{\partial V}{\partial r}\right)\vec{s}\cdot\vec{\ell} \\
\vec{s} &= \frac{\hbar}{2}\vec{\sigma} \\
\vec{\ell} &= \vec{r}\times\vec{p}
\end{aligned}
$$

and is called the spin-orbit interaction.

The Time-dependent Case (The Lowest Order)

Given

$$
\Psi = e^{-imc^2t/\hbar}\begin{pmatrix}\psi \\ \chi\end{pmatrix}
$$

Recall our discussion for the steady states, and we direct our attention to the slow mode in energy mc^2 periphery:

$$
\begin{aligned}
mc^2\psi + i\hbar\partial_t\psi &= (mc^2 + e\phi)\psi + cP\chi \\
mc^2\chi + i\hbar\partial_t\chi &= cP\psi + (-mc^2 + e\phi)\chi
\end{aligned}
$$

which gives

$$
\begin{aligned}
i\hbar\partial_t\psi &= e\phi\psi + cP\chi \\
i\hbar\partial_t\chi &= cP\psi + (-2mc^2 + e\phi)\chi
\end{aligned}
$$

We define $mv^2 << mc^2$, $e\phi << mc^2$, the second equation may give

$$
\chi = \frac{cP}{2mc^2}\psi
$$

Thus, we can derive the Schr?dinger equation

$$
\begin{aligned}
i\hbar\frac{\partial\psi}{\partial t} &= H_{sh}\psi \\
H_{sh} &= \frac{P^2}{2m} = \frac{(\vec{p}-e\vec{A})^2}{2m} + e\phi + \vec{\mu}\cdot\vec{B}
\end{aligned}
$$

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$$
\vec{E} = -\frac{\partial V}{\partial r}\hat{r} = -\frac{1}{r}\frac{\partial V}{\partial r}\vec{r}
$$