Quantum Mechanics 2: Angular Momentum Composition

Spin angular momentum of electrons is, quantum mechanically, the internal degree of freedom, which is different from orbital angular momentum. Spin takes on the values of $(1/2)\hbar$, and its matrix components are mentioned in the generalization of angular momentum. Let's now consider an electron with $\ell = 1, s = 1/2$.

(1) Write out the matrix components of ℓ^2, ℓ_z and s^2, s_z for the arbitrary basic function.

(2) Investigate the orbital angular momentum and spin angular momentum at the same time.

There are six basis for the eigenfunction for the base $|\ell, m_\ell; s, m_s\rangle \equiv |m_\ell; m_s\rangle$, $|1;1/2\rangle$, $|1;-1/2\rangle$, $|0;1/2\rangle$, $|0;-1/2\rangle$,$|-1;1/2\rangle$, $|-1;-1/2\rangle$ these are the six bases .

Now, we calculate the matrix component of $\vec{l}^2, j_z$ by defining $\vec{j} = \vec{\ell} + \vec{s}, j_z = \ell_z + s_z, j_\pm = \ell_\pm + s_\pm$ to the basis. Then we determine the basis vectors that transform these two matrixes to be diagonal. If it is completed, we can express the basis vectors in terms of $|\ell, m_\ell; s, m_s\rangle$ to finally determine $j, m_j, \ell, m_\ell, s, m_s$.

First, we consider (1)(2). Take following basis functions to write out the matrix component:

$$|m_l, m_s\rangle = |1, \frac{1}{2}\rangle,\ |1, -\frac{1}{2}\rangle,\ |0, \frac{1}{2}\rangle,\ |0, -\frac{1}{2}\rangle,\ |-1, \frac{1}{2}\rangle,\ |-1, -\frac{1}{2}\rangle$$

(Set an unit $\hbar = 1$)

$$l^2 = \hbar^2 \begin{pmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix}, \quad l_z = \hbar \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix}$$

$$s^2 = \hbar^2 \begin{pmatrix} \frac{3}{4} & & & & & \\ & \frac{3}{4} & & & & \\ & & \frac{3}{4} & & & \\ & & & \frac{3}{4} & & \\ & & & & \frac{3}{4} & \\ & & & & & \frac{3}{4} \end{pmatrix}, \quad s_z = \hbar \begin{pmatrix} \frac{1}{2} & & & & & \\ & -\frac{1}{2} & & & & \\ & & \frac{1}{2} & & & \\ & & & -\frac{1}{2} & & \\ & & & & \frac{1}{2} & \\ & & & & & -\frac{1}{2} \end{pmatrix}$$

Based on the equation: $\vec{l} \cdot \vec{s} = l_x s_x + l_y s_y + l_z s_z = \frac{1}{2}(l_+ s_- + l_- s_+) + l_z s_z$:

$(l \cdot s)|1\frac{1}{2}\rangle = \frac{\hbar^2}{2}|1\frac{1}{2}\rangle$ can be established, thus:

$$l \cdot s = \hbar^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

So,

$$j^2 = (l+s)^2 = l^2 + s^2 + 2l \cdot s = \hbar^2(2 + \frac{3}{4})\mathbf{1} + \hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Conduct diagonalization of matrix in the right hand second term:

Eigenvalue of j^2:

$$(2 + \frac{3}{4} - 2)\hbar^2 = \frac{3}{4}\hbar^2 \ : \ (j = \frac{1}{2}) \text{ double}$$

$$(2 + \frac{3}{4} + 1)\hbar^2 = \frac{15}{4}\hbar^2 \ : \ (j = \frac{3}{2}) \text{ quadruple}$$

For eigenfunction:

For $j = \frac{3}{2}$:

$$|a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |a_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$|a_3\rangle = \cos\theta \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} \equiv \cos\theta|A\rangle + \sin\theta|B\rangle,$$

$$|a_4\rangle = -\sin\theta \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \cos\theta \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} \equiv -\sin\theta|A\rangle + \cos\theta|B\rangle$$

For $j = \frac{1}{2}$:

$$|b_1\rangle = \cos\phi \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sin\phi \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix} = \cos\phi|C\rangle + \sin\phi|D\rangle,$$

$$|b_2\rangle = -\sin\phi \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \cos\phi \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix} = -\sin\phi|C\rangle + \cos\phi|D\rangle$$

$$j_z|a_1\rangle = \frac{3}{2}\hbar|a_1\rangle, j_z|a_2\rangle = -\frac{3}{2}\hbar|a_2\rangle, j_z|A\rangle = \frac{1}{2}\hbar|A\rangle,$$

$$j_z|B\rangle = -\frac{1}{2}\hbar|B\rangle, j_z|C\rangle = \frac{1}{2}\hbar|C\rangle, j_z|D\rangle = -\frac{1}{2}\hbar|D\rangle$$

Therefore, we can obtain the following bases that diagonalize j^2 and j_z at the same time:

$$j = \frac{3}{2} : \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$j = \frac{1}{2} : \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ \sqrt{2} \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

Vector composition: Angular momentum composition

$$\mathbf{j} = \vec{\ell} + \mathbf{s}$$

$$|\ell s : j m_j >= \sum_{m_\ell m_s} |\ell m_\ell : s m_s > \langle \ell m_\ell : s m_s | j m_j \rangle$$

Wingner coefficients, Clebsch-Gordan coefficients:

$$\langle J_1 M_1 J_2 M_2 | JM \rangle = \delta(M, M_1 + M_2)\sqrt{2J+1}\triangle(j_1 J_2 J)$$
$$\times \sqrt{(J_1 + M_1)!(J_1 - M_1)!(J_2 + M_2)!(J_2 - M_2)!(J + M)(J - M)!}$$
$$\times \sum_z (-1)^Z [Z!(J_1 + J_2 - J - Z)!(J_1 - M_1 - Z)!(J_2 + M_2 - Z)!$$
$$\times (J - J_2 + M_1 + Z)!(J - J_1 - M_2 + Z)!]^{-1}$$

$$\triangle(J_1 J_2 J) = \sqrt{\frac{(J_1 + J_2 - J)!(J + J_1 - J_2)!(J + J_2 - J_1)!}{(J_1 + J_2 + J + 1)!}}$$