Relativistic Electron Theory

Dirac Equation

Relativistic equation:

$$E^2 - p^2c^2 = m^2c^4$$

Define the condition for the equation above:

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \ , \ p_x \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}$$

Form a differential equation that makes the first-order differentiation in time:

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \ , \ H = c\vec{\alpha}\cdot\boldsymbol{p} + \beta mc^2 \ ; \ \vec{\alpha} \ , \ \beta = \text{Usually Matrix}$$

$$(i\hbar\frac{\partial}{\partial t} - c\vec{\alpha}\cdot\boldsymbol{p} - \beta mc^2)\psi = 0$$

Operate on the conjugate operator from the left hand side:

$$(i\hbar\frac{\partial}{\partial t} + c\vec{\alpha}\cdot\boldsymbol{p} + \beta mc^2)(i\hbar\frac{\partial}{\partial t} - c\vec{\alpha}\cdot\boldsymbol{p} - \beta mc^2)\psi = 0$$

The following conditions for $\vec{\alpha}, \beta$ are then obtained through the comparison of the equation above and the equation $E^2 - p^2c^2 - m^2c^4 = 0$.

$$\begin{cases} \alpha_k\alpha_\ell + \alpha_\ell\alpha_k = 2\delta_{k\ell} \\ \alpha_k\beta + \beta\alpha_k = 0 \\ \beta^2 = 1 \end{cases}$$

To satisfy the condition, $\vec{\alpha}, \beta$ should be 4×4 matrix, and ψ is the vector in four-dimention. (from now on, $\vec{\sigma}$ represents Pauli matrix)

$$\beta = \begin{pmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{1} \end{pmatrix} \ , \ \vec{\alpha} = \begin{pmatrix} \boldsymbol{0} & \vec{\sigma} \\ \vec{\sigma} & \boldsymbol{0} \end{pmatrix}$$

$$\boldsymbol{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ \boldsymbol{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Dirac\ equation \ : \ (i\hbar\frac{\partial}{\partial t} - c\vec{\alpha}\cdot\boldsymbol{p} - \beta mc^2)\psi = 0 \ , \ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

probability: $P = \psi^\dagger\psi = \sum_i |\psi_i|^2$

electric current density: $\boldsymbol{j} = \psi^\dagger\vec{\alpha}\psi$

Free electron:

$$
\begin{aligned}
&\psi_j = u_j e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}\\
&E_\pm = \pm\sqrt{c^2p^2+m^2c^4}\\
&E=E_+>0: \qquad u_+^{(1)} \sim \begin{pmatrix} 1 \\ 0 \\ cp_z/(E_+ + mc^2) \\ c(p_x+ip_y)/(E_+ + mc^2) \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ v/c \\ v/c \end{pmatrix}\\
&\qquad\qquad\qquad\quad u_+^{(2)} \sim \begin{pmatrix} 0 \\ 1 \\ c(p_x-ip_y)/(E_+ + mc^2) \\ -cp_z/(E_+ + mc^2) \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ v/c \\ v/c \end{pmatrix}\\
&E=E_-<0: \qquad u_-^{(1)} \sim \begin{pmatrix} -cp_z/(-E_- + mc^2) \\ -c(p_x+ip_y)/(-E_- + mc^2) \\ 1 \\ 0 \end{pmatrix}\\
&\qquad\qquad\qquad\quad u_-^{(2)} \sim \begin{pmatrix} -c(p_x-ip_y)/(-E_- + mc^2) \\ cp_z/(-E_- + mc^2) \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

Free electron in electromagnetic field

$$\{i\hbar\frac{\partial}{\partial t} + e\phi - c\vec{\alpha}\cdot(\boldsymbol{p}+e\boldsymbol{A}) - \beta mc^2\}\psi = 0$$

Where $E>0$,

$$\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} = \frac{c\vec{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A})}{E+e\phi+mc^2}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$E+e\phi+mc^2 \approx 2mc^2\ ,\ \vec{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A})\ \vec{\sigma}\cdot(\boldsymbol{p}+e\boldsymbol{A}) = (\boldsymbol{p}+e\boldsymbol{A})^2 + e\hbar\vec{\sigma}\cdot\boldsymbol{H}$$

$$\{\frac{1}{2m}(\boldsymbol{p}+e\boldsymbol{A})^2 + \frac{e\hbar}{2m}\vec{\sigma}\cdot\boldsymbol{H} - e\phi\}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = E\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Central force field

$$
\begin{aligned}
&H = c\vec{\alpha}\cdot\boldsymbol{p} + \beta mc^2 + V(r)\\
&E' = E - mc^2\\
&[\frac{1}{2m(1+\frac{E'-V}{2mc^2})}\boldsymbol{p}^2 + V - \frac{\hbar^2}{4m^2c^2}\frac{dV}{dr}\frac{\partial}{\partial r} + \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{\ell}\cdot\vec{s}]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = E'\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}
\end{aligned}
$$