Orbital Angular Momentum: Symmetry and Conservation (cont.2)

Algebraic Handling for Eigenstate of Angular Momentum

The relational equations for the eigenfunction of angular momentum (35)(40) are expressed with specific formations of eigenfunction. The derivation of an equation (40) is considered relatively difficult. Indeed, these equations can be solved solely by the commutation relations (41a~c) without any knowledge for specific formations of the eigenfunction. In order to explain this fact, we take eigenfunction as:

$$\hat{\ell}^2\varphi_{jm} = \hbar^2\lambda_j\varphi_{jm}, \qquad \hat{\ell}_z\varphi_{jm} = \hbar m\varphi_{jm} \tag{42}$$

To derive (27), we used only $\hat{\ell}^2 = \hat{\ell}_x^2 + \hat{\ell}_y^2 + \hat{\ell}_z^2$, and Hermitian of $\hat{\ell}_x, \hat{\ell}_y$ as a tool. Thus, in the same manner, we can describe the following for (42):

$$\lambda_j \geq m^2 \geq 0$$

In other words, an inclusion of the upper and lower limit of m is found. In the next step, the commutator $[\hat{\ell}_z, \hat{\ell}_\pm] = \pm\hbar\hat{\ell}_\pm$ of (41b) is operated upon φ_{jm},

$$\hat{\ell}_z\hat{\ell}_\pm\varphi_{jm} = \hat{\ell}_\pm\hat{\ell}_z\varphi_{jm} \pm \hbar\hat{\ell}_\pm\varphi_{jm} = \hbar(m\pm1)\hat{\ell}_\pm\varphi_{jm}$$

We can say that $\hat{\ell}_\pm\varphi_{jm}$ takes the eigenfunction that belongs to the eigenvalue $\hbar(m\pm1)$ of $\hat{\ell}_z$. To describe this:

$$\hat{\ell}_\pm\varphi_{jm} = C_{jm}^{(\pm)}\varphi_{jm\pm1} \tag{43}$$

With a fixed λ_j, the upper limit of m we write m_U, and the lower limit: m_L, then it should be:

$$\hat{\ell}_+\varphi_{jm_U} = 0, \qquad \hat{\ell}_-\varphi_{jm_L} = 0 \tag{44}$$

Conduct an operation for the first equation on $\hat{\ell}_-$, and for the latter equation, conduct an operation on $\hat{\ell}_+$, also, to transform the equation (41c) by using (42) then:

$$\hat{\ell}_-\hat{\ell}_+\varphi_{jm_U} = (\hat{\ell}^2 - \hat{\ell}_z^2 - \hbar\hat{\ell}_z)\varphi_{jm_U} = \hbar^2(\lambda_j - m_U^2 - m_U)\varphi_{jm_U} = 0,$$

$$\hat{\ell}_+\hat{\ell}_-\varphi_{jm_L} = (\hat{\ell}^2 - \hat{\ell}_z^2 + \hbar\hat{\ell}_z)\varphi_{jm_L} = \hbar^2(\lambda_j - m_L^2 + m_L)\varphi_{jm_L} = 0$$

So, we can say:

$$\lambda_j = m_U^2 + m_U = m_L^2 - m_L \tag{45}$$

From (43), the possible values of m should be the numbers with the difference 1, (we have not yet defined them as integer) hence:

$$m_U - m_L = \text{positive integer or 0}$$

With the equation (45), the following is derived:

$$m_U = -m_L \geq 0$$

What the equation above implies is the fact m takes a half integer or 0. To write the upper limit m_U as j, then $2j$ takes either 0 or positive integer, therefore:

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \qquad (46)$$

From the equation (45):

$$\lambda_j = j(j+1) \qquad (47)$$

In comparison with (47) to (35a), we can find what is described by j in here, corresponds to what we have written in ℓ.

Moreover, the inner product of $\langle \hat{\ell}_\pm \varphi_{jm} | \hat{\ell}_\pm \varphi_{jm} \rangle$ can be calculated by using (43):

$$\begin{aligned} \langle \hat{\ell}_\pm \varphi_{jm} | \hat{\ell}_\pm \varphi_{jm} \rangle &= |C_{jm}^{(\pm)}|^2 \langle \varphi_{jm\pm1} | \varphi_{jm\pm1} \rangle = |C_{jm}^{(\pm)}|^2 \\ &= \langle \varphi_{jm} | \hat{\ell}_\mp \hat{\ell}_\pm \varphi_{jm} \rangle = \langle \varphi_{jm} | (\hat{\ell}^2 - \hat{\ell}_z^2 \mp \hbar \hat{\ell}_z) \varphi_{jm} \rangle \\ &= \langle \varphi_{jm} | \hbar^2 (j(j+1) - m^2 \mp m) \varphi_{jm} \rangle = \hbar^2 (j \mp m)(j \pm m + 1). \end{aligned}$$

Thus, the coefficient of (43) is defined as:

$$C_{jm}^{(\pm)} = \hbar \sqrt{(j \mp m)(j \pm m + 1)} \qquad (48)$$

The phase factor is yet to be defined, though we usually put 1.

By following these steps, relationships such as (27), (35a), (35b), (40) are defined, however, it is important to be aware of the fact that in the discussions above, the value of ℓ (here expressed as j) are defined as 0, integer or a half odd integer, thus the integer is not necessarily taken, as we can see in (46). We can intuitively learn of the value ℓ when it is a half odd integer, for example, the wavefunction cannot take univalence spatially, by picturing the images of a specific wavefunction. An example for j taking a half odd integer can be $j = 1/2$ spin.

Invariant for Infinitesimal Transformation

As we have learned, Hamiltonian is related to the time progress operator, and the invariant for the operator can be considered as energy. In the same way, conservation of momentum and conservation of angular momentum are linked to the symmetry of general systems

----Fig.6-3-------

First, let's consider for a new coordinate system, in which the coordinate axis is conducted a parallel translation by $\delta \boldsymbol{r}$. (fig.6.3a) An old point for $\boldsymbol{r}$ moved to:

$$\boldsymbol{r}' = \boldsymbol{r} - \delta \boldsymbol{r} \tag{49}$$

The momentum of both coordinate systems remains constant:

$$\boldsymbol{p}' = \boldsymbol{p} \tag{49'}$$

We write the wavefunction in an old coordinate system as ψ, and the one in a new coordinate system as ψ'. Since $\boldsymbol{r}$ and $\boldsymbol{r}'$ represents the same points, the values for the wavefunction supposed to remain constant even after the coordinate translation is conducted:

$$\psi'(\boldsymbol{r}') = \psi(\boldsymbol{r}) = \psi(\boldsymbol{r}' + \delta \boldsymbol{r}).$$

Or rather, rewrite $\boldsymbol{r}'$ in the form $\boldsymbol{r}$, and operate Tayler's expansion in terms of $\delta \boldsymbol{r}$:

$$\begin{aligned}\psi'(\boldsymbol{r}) &= \psi(\boldsymbol{r} + \delta \boldsymbol{r}) \cong \psi(\boldsymbol{r}) + \delta \boldsymbol{r} \cdot \nabla \psi(\boldsymbol{r}) = \psi(\boldsymbol{r}) + \frac{i}{\hbar} \delta \boldsymbol{r} \cdot \hat{\boldsymbol{p}} \psi(\boldsymbol{r}) \\ &= (1 + \frac{i}{\hbar} \delta \boldsymbol{r} \cdot \hat{\boldsymbol{p}}) \psi(\boldsymbol{r})\end{aligned} \tag{50}$$

The equation above refers to the operator that represents the infinitesimal coordinate transformation of (49) to be the following:

$$U_{\delta \boldsymbol{r}} = 1 + \frac{i}{\hbar} \delta \boldsymbol{r} \cdot \hat{\boldsymbol{p}} \tag{51}$$

In correspondence to the coordinate transformation, Hamiltonian is also transformed to be:

$$H' = U_{\delta \boldsymbol{r}} H U_{\delta \boldsymbol{r}}^{-1} = H - \frac{i}{\hbar} [H, \delta \boldsymbol{r} \cdot \hat{\boldsymbol{p}}] + O(\delta \boldsymbol{r}^2) \tag{52}$$

In accordance with the infinitesimal parallel transformation, we can conclude that Hamiltonian is invariable against the event: $H = H'$, and the displacement directional component of the momentum $\hat{\boldsymbol{p}}$ is transformed with Hamiltonian that is equivalance. Inversely, if there was an existence of the potential within a system that can possibly break its uniformity, the momentum will not be conserved because the particles are scattered and the momentum is exchanged. $([H, \hat{\boldsymbol{p}}] \neq 0)$

Now, we consider a new coordinate system, in which the coordinate axis is rotated about a fixed unit vector $\boldsymbol{e}(|\boldsymbol{e}| = 1)$ by $\delta\theta$. (fig.6.3b) The point $\boldsymbol{r}$ from an old coordinate system can be found in the new coordinate system at:

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{r} \times \boldsymbol{e}\delta\theta = \boldsymbol{r} - \boldsymbol{r}' \times \boldsymbol{e}\delta\theta \tag{53}$$

Also,

$$\psi'(\boldsymbol{r}') = \psi(\boldsymbol{r}) = \psi(\boldsymbol{r}' + \boldsymbol{r}' \times \boldsymbol{e}\delta\theta)$$

For Tayler's expansion:

$$\psi'(\boldsymbol{r}) = \psi(\boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{e}\delta\theta) \cong (1 + \delta\theta \boldsymbol{r} \times \boldsymbol{e} \cdot \nabla)\psi(\boldsymbol{r})$$
$$= (1 - \delta\theta \boldsymbol{e} \cdot (\boldsymbol{r} \times \nabla))\psi(\boldsymbol{r}) = (1 - \frac{i}{\hbar}\delta\theta \boldsymbol{e} \cdot \hat{\ell})\psi(\boldsymbol{r}) \qquad (54)$$

The angular momentum operator $\hat{\ell} = \boldsymbol{r} \times \hat{\boldsymbol{p}}$ is found in the equation, which implies the operator with infinitesimal rotation:

$$U_{\delta\theta} = 1 - \frac{i}{\hbar}\delta\theta \boldsymbol{e} \cdot \hat{\ell} \qquad (55)$$

For the infinitesimal transformation $U_{\delta\theta}$, Hamiltonian is:

$$H' = U_{\delta\theta} H U_{\delta\theta}^{-1} = H + \frac{i}{\hbar}[H, \delta\theta \boldsymbol{e} \cdot \hat{\ell}] + O(\delta\theta^2) \qquad (56)$$

Therefore, when the system holds the invariance of Hamiltonian at infinitesimal rotation: $H' = H$, we can also conclude that component of angular momentum $\boldsymbol{e} \cdot \hat{\ell}$ toward its axis direction and Hamiltonian can be exchanged. These two facts are understood to be identical.