Orbital Angular Momentum: Symmetry and Conservation (cont.)

Conservation of Orbital Angular Momentum (The momentum for a single particle in the central force field)

The central force is the "conservative force", and thus the potential energy can be defied.

Where the vector of the force is $\boldsymbol{F}(\boldsymbol{r})$,

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z} \qquad (12)$$

We can define a univalent scalar function $V(\boldsymbol{r})$ (potential function, potential energy) about a position vector $\boldsymbol{r}$. For the central force field, $V(\boldsymbol{r})$is a function exclusively related to the distance from the origin $r = |\boldsymbol{r}|$, that is $V(\boldsymbol{r}) = V(r)$. At this moment, Hamiltonian for a particle may be:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(r) \qquad (13)$$

By using the relational functions (6a~b)(11a~b), we can establish the following for the Hamiltonian (13):

$$[\hat{\ell}_\alpha, H] = 0, \quad [\hat{\ell}^2, H] = 0 \quad (\alpha = x, y, z) \qquad (14)$$

$\hat{\ell}_\alpha$is operated with a kinetic energy term $(\hbar^2/2m)\Delta$■ in (11a~b), and in (6a~b), $\hat{\ell}_\alpha$is operated with a spherically symmetric potential term $V(r)$.

From what we have established in (14), each component of the orbital angular momentum operators $\hat{\ell}_x$、$\hat{\ell}_y$、$\hat{\ell}_z$ are operated with the Hamiltonian, hence we can define an eigenfunction of the Hamiltonian as to match with the eigenfunction of the orbital angular momentum operators $\hat{\ell}^2$ or either one of $\hat{\ell}_x$、$\hat{\ell}_y$、$\hat{\ell}_z$. This is the law of conservation of orbital angular momentum from quantum mechanical point of view. As we can find in (10), each component of the orbital angular momentum $\hat{\ell}_x$、$\hat{\ell}_y$、$\hat{\ell}_z$ does not operate with one another, if the energy eigenfunction is picked to match with the certain eigenfunctions $\hat{\ell}^2$and$\hat{\ell}_z$ simultaneously, there is no way that we can take the eigenfunctions of $\hat{\ell}_x$ or $\hat{\ell}_y$. In general, the energy eigenfunction may be taken to match with the eigenfunction of $\hat{\ell}^2$and$\hat{\ell}_z$.

Figure.6-1(polar coordinates: abbr.)

Orbital Angular Momentum in Polar Coordinates

It is possible to further the discussion on the orbit angular momentum operator without using any representations of θ, ϕ, however, in this section, we write off only the results represented with θ, ϕ. The polar coordinates r, θ, ϕ are defined as shown in the Fig.6.1. In formula, we can write as:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi \qquad (15)$$
$$z = r\cos\theta$$

Or inversely,

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2}, \\ \cos\theta &= z/\sqrt{x^2+y^2+z^2}, \\ \tan\phi &= y/x, \\ 0 \le \theta &\le \pi, \quad 0 \le \phi < 2\pi \end{aligned} \qquad (15')$$

Based on these formulas, we can conduct polar coordinates transformation about (5):

$$\begin{aligned} \hat{\ell}_x &= i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}), \\ \hat{\ell}_y &= i\hbar(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}), \\ \hat{\ell}_z &= -i\hbar\frac{\partial}{\partial\phi}, \\ \hat{\ell}^2 &= -\hbar^2[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}] \end{aligned} \qquad (16)$$

We leave this calculation of the variable transformation up to the readers.

Eigenfunction of Orbital Angular Momentum Operators $\hat{\ell}^2$ and $\hat{\ell}_z$

As a first step, we try to obtain the eigenfunction of $\hat{\ell}^2$. We define $\varphi_\ell(x, y, z)$ as, harmonic function (we call harmonic polynomial) of the ℓ degrees homogeneous polynomial of x, y, z. Harmonic function is the function that satisfies following:

$$\Delta\varphi = 0 \qquad (17)$$

In this occasion, φ_ℓ can be expressed as:

$$\varphi_\ell(x, y, z) = \sum_{n_x, n_y, n_z \ge 0 \quad (n_x+n_y+n_z=\ell)}^{\ell} a_{n_x n_y n_z} x^{n_x} y^{n_y} z^{n_z}. \qquad (18)$$

In polar coordinates, the formula can be reformed as:

$$\varphi_\ell(x,y,z) = r^\ell \psi_\ell(\theta,\phi) \qquad (19)$$

By conducting Laplacian operation, and taking into account that the expression (9') and φ_ℓ are the harmonic functions, we can get:

$$\Delta\varphi_\ell(x,y,z) = \frac{\ell(\ell+1)}{r^2}\varphi_\ell - \frac{1}{r^2\hbar^2}\hat{\ell}^2\varphi_\ell = 0 \qquad (20)$$

Therefore, the harmonic polynomial function is considered to be the eigenfunction of $\hat{\ell}^2$ with an eigenvalue $\hbar^2\ell(\ell+1)$.

$$\hat{\ell}^2\varphi_\ell(x,y,z) = \hbar^2\ell(\ell+1)\varphi_\ell(x,y,z). \qquad (21)$$

We emphasize again, that $\hat{\ell}^2$ does not include $(\partial/\partial r)$.

When $\hat{\ell}_z$ holds the eigenfunction $\varphi_{\ell m}$ with an eigenvalue $\hbar m$, and given that $\hat{\ell}_z = -i\hbar(\partial/\partial\phi)$ as stated in (16), we can write as:

$$\hat{\ell}_z\varphi_{\ell m}(x,y,z) = -i\hbar\frac{\partial}{\partial\phi}\varphi_{\ell m}(x,y,z) = \hbar m\varphi_{\ell m}(x,y,z) \qquad (22)$$

The differential equation (22) can be easily solved despite that it is the function of ϕ, in which $\varphi_{\ell m} \sim e^{im\phi}$, with a reference to (19), we can obtain:

$$\varphi_{\ell m}(x,y,z) = r^\ell N_{\ell m}\frac{e^{im\phi}}{\sqrt{2\pi}}P_{\ell m}(\cos\theta) \qquad (23)$$

The above applies only if $N_{\ell m}$ was a constant number, and $P_{\ell m}(\cos\theta)$ was an angle θ, which being defined upon dependence of ℓ and m. θ is limited within the region $\pi \geq \theta \geq 0$, thus, there should be no confusion over the ways of expressing the angle θ or $\cos\theta$.

Now, let's consider over the polynomial function $\varphi_{\ell m}(x,y,z)$. The zero degree polynomial function is apparently:

$$\varphi_{00}(x,y,z) = \frac{1}{\sqrt{4\pi}} \qquad (24)$$

For the first-degree homogeneous expression, there exists the following three:

$$\begin{aligned}\varphi_{1x} &= \frac{\sqrt{3}}{2\sqrt{\pi}}x,\\ \varphi_{1y} &= \frac{\sqrt{3}}{2\sqrt{\pi}}y,\\ \varphi_{1z} &= \frac{\sqrt{3}}{2\sqrt{\pi}}z\end{aligned} \qquad (25)$$

which are the solutions for the Laplace's equation $\Delta\varphi = 0$, also they are what should be obtained for $\ell = 1$. As for φ_{1z}, it is consisted of the eigenfunction that satisfies $\hbar m = 0$ for $\hat{\ell}_z$, and satisfies the following condition:

$$\hat{\ell}_z \varphi_{1z} = 0$$

On the other hand, φ_{1x} and φ_{1y} may be stated as:

$$\hat{\ell}_z \varphi_{1x} = -\frac{\hbar}{i}\varphi_{1y}, \quad \hat{\ell}_z \varphi_{1y} = \frac{\hbar}{i}\varphi_{1x}$$

That are not consisted of the eigenfunction of $\hat{\ell}_z$, yet the two expressions above when combined together would be:

$$\hat{\ell}_z(\varphi_{1x} \pm i\varphi_{1y}) = (\pm\hbar)(\varphi_{1x} \pm i\varphi_{1y})$$

Indicating that each and every value of φ may be consisting the eigenfunction with eigenvalue $\pm\hbar$ of $\hat{\ell}_z$. In this way:

$$\begin{aligned}
\varphi_{11} &= -\frac{1}{\sqrt{2}}(\varphi_{1x} + i\varphi_y) = -\sqrt{\frac{3}{8\pi}}(x+iy) = -\sqrt{\frac{3}{8\pi}} r \sin\theta e^{i\phi}, \\
\varphi_{10} &= \varphi_{1z} = \sqrt{\frac{3}{4\pi}} z = \sqrt{\frac{3}{4\pi}} r\cos\theta, \qquad (25') \\
\varphi_{1-1} &= \frac{1}{\sqrt{2}}(\varphi_{1x} - i\varphi_{1y}) = \sqrt{\frac{3}{8\pi}}(x-iy) = \sqrt{\frac{3}{8\pi}} r\sin\theta e^{-i\phi}
\end{aligned}$$

Each value of φ is the eigenfunction that takes eigenvalues $\ell = 1$、$m = +1, 0, -1$. In defining φ_{11}, there is not much of significance in the meanings for making a negative multiplication to the equation for now.

Although, when $\ell = 2$, the homogeneous polynomials in the second degree are found to be x^2, y^2, z^2, yz, zx, xy, all of them would not necessarily turn out to be the solution for the Laplace's equation. Given $x^2 + y^2 + z^2 = r^2$, among the six of the second-degree polynomials, five linear combinations of those can be the independent eigenfunction of $\ell = 2$. In most of the times, the following five linear combinations are chosen:

$$\begin{aligned}
\varphi_{2,3z^2-r^2} &= \sqrt{\frac{5}{16\pi}}(3z^2-r^2) = \sqrt{\frac{5}{16\pi}}r^2(3\cos^2\theta-1),\\
\varphi_{2,x^2-y^2} &= \sqrt{\frac{15}{16\pi}}(x^2-y^2) = \sqrt{\frac{15}{16\pi}}r^2\sin^2\theta\cos 2\phi,\\
\varphi_{2,yz} &= \sqrt{\frac{15}{4\pi}}yz = \sqrt{\frac{15}{4\pi}}r^2\sin\theta\cos\theta\sin\phi,\\
\varphi_{2,zx} &= \sqrt{\frac{15}{4\pi}}zx = \sqrt{\frac{15}{4\pi}}r^2\sin\theta\cos\theta\cos\phi,\\
\varphi_{2,xy} &= \sqrt{\frac{15}{4\pi}}xy = \sqrt{\frac{15}{4\pi}}r^2\sin^2\theta\sin 2\phi.
\end{aligned} \tag{26}$$

These are the solutions for the Laplace's equation, and it is easy to notice the eigenfunction of $\hat{\ell}^2$, which correspond with $\ell = 2$. Although, they are not the eigenfunction of $\hat{\ell}_z$, as it was for $\ell = 1$, we can still obtain the eigenfunctions for $m = \pm 2$ and $m = \pm 1$ respectively by establishing the linear combination of φ_{2,x^2-y^2}, $\varphi_{2,xy}$, $\varphi_{2,yz}$, and $\varphi_{2,zx}$. In rewriting the eigenfunction in the actual form of $\hat{\ell}_z$:

$$\begin{aligned}
\varphi_{22} &= \frac{1}{\sqrt{2}}(\varphi_{2,x^2-y^2} + i\varphi_{2,xy}) = \frac{\sqrt{15}}{4\sqrt{2\pi}}r^2\sin^2\theta e^{2i\phi},\\
\varphi_{21} &= -\frac{1}{\sqrt{2}}(\varphi_{2,zx} + i\varphi_{2,yz}) = -\frac{\sqrt{15}}{2\sqrt{2\pi}}r^2\sin\theta\cos\theta e^{i\phi},\\
\varphi_{20} &= \varphi_{2,3z^2-r^2} = \frac{\sqrt{5}}{2\sqrt{4\pi}}r^2(3\cos^2\theta-1),\\
\varphi_{2-1} &= \frac{1}{\sqrt{2}}(\varphi_{2,zx} - i\varphi_{2,yz}) = \frac{\sqrt{15}}{2\sqrt{2\pi}}r^2\sin\theta\cos\theta e^{-i\phi},\\
\varphi_{2-2} &= \frac{1}{\sqrt{2}}(\varphi_{2,x^2-y^2} - i\varphi_{2,xy}) = \frac{\sqrt{15}}{4\sqrt{2\pi}}r^2\sin^2\theta e^{-2i\phi}
\end{aligned} \tag{26'}$$

Based on the results showing above, where $\ell = 0, 1, 2$, the eigenfunction of the angular momentum $\hbar\ell$ exists as many as $2\ell+1$, and each one of them corresponds $m = \ell, \ell-1, \cdots, -\ell+1, -\ell$. Which indicates that it is also established at any ℓ. To derive $\ell \geq |m|$, we take the following steps:

Using the Hermitian of the following that are equally established, so we can derive:

$$\int \{\varphi^*_{\ell m}\hat{\ell}^2_\alpha \varphi_{\ell m}\}\sin\theta d\theta d\phi = \int \{|\hat{\ell}_\alpha \varphi_{\ell m}|^2\}\sin\theta d\theta d\phi \geq 0$$

$$\begin{aligned}
&\int \{\varphi^*_{\ell m}(\hat{\ell}^2 - \hat{\ell}^2_z)\varphi_{\ell m}\}\sin\theta d\theta d\phi\\
&\qquad = (\ell(\ell+1) - m^2)\hbar^2 \int |\varphi_{\ell m}|^2 \sin\theta d\theta d\phi \geq 0
\end{aligned}$$

$\ell(\ell+1) \geq m^2$, and assume m to be the integer then:

$$\ell \geq m \geq -\ell \qquad (27)$$

For the appropriate selection of the constant numbers that are shown at the top of (25)~(26'), and $N_{\ell m}$ of (24), it is common and advantageous to normalize the wavefunctions of square-integrable measurable functions to be 1:

$$\int |\varphi_{\ell m}(x,y,z)|^2 \sin\theta d\theta d\phi = N_{\ell m}^2 r^{2\ell} \int_0^\pi \sin\theta d\theta |P_{\ell m}(\cos\theta)|^2 = r^{2\ell} \qquad (28)$$

Spherical Functions (Abbrev. in lectures)

Through (25)~(26'), we have deepen our understanding over the angular momentum operator $\hat{\ell}^2$, as well as the eigenfunction $\varphi_{\ell m}$ of $\hat{\ell}_z$. To consider them in more generalized terms, it is possible to take (21) as differential equations in terms of θ. Moreover, the expression for $\hat{\ell}^2$ given in (16) and (23) clarifies the conditions for $P_{\ell m}(\cos\theta)$ to satisfy the following differential equations:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP_{\ell m}(\cos\theta)}{d\theta}) + (\ell(\ell+1) - \frac{m^2}{\sin^2\theta})P_{\ell m}(\cos\theta) = 0. \quad (29)$$

To conduct the variable transformation with $\omega = \cos\theta$, we can also write as:

$$\frac{d}{d\omega}[(1-\omega^2)\frac{dP_{\ell m}(\omega)}{d\omega}] + (\ell(\ell+1) - \frac{m^2}{1-\omega^2})P_{\ell m}(\omega) = 0 \qquad (29')$$

The differential equation showing above is well scrutinized over a period of time that it is called the associated Legendre's differential equations, and its solution $P_{\ell m}(\omega)$is called an associated Legendre function. As we have analyzed in (27), ℓ takes either 0 or the positive integer, and m takes either 0 or positive and negative integer within the region $\ell \geq m \geq -\ell$.

In (29'), where $m = 0$, the differential equation can be:

$$\frac{d}{d\omega}[(1-\omega^2)\frac{dP_\ell(\omega)}{d\omega}] + \ell(\ell+1)P_\ell(\omega) = 0 \qquad (30)$$

The differential equation showing above is called Legendre's differential equations, and its solution is called Legendre function. $P_\ell(\omega)$ takes ℓ-degree polynomials about ω, where ℓ is the integer or 0, so we can write as:

$$P_\ell(\omega) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{d\omega^\ell}(\omega^2-1)^\ell = \frac{(-1)^\ell}{2^\ell \ell!}\frac{d^\ell}{d\omega^\ell}(1-\omega^2)^\ell \qquad (31)$$

To make specific for the formations when $\ell = 0, 1, 2, 3$,

$$P_0(\omega) = 1$$

$$P_1(\omega) = \omega$$

$$P_2(\omega) = (1/2)(3\omega^2 - 1)$$

$$P_3(\omega) = (1/2)(5\omega^3 - 2\omega)$$

The coefficients $(-1)^\ell/(2^\ell \ell!)$ are arbitrarily decided.

Generally, the associated Legendre function $P_{\ell m}(\omega)$ is expressed by using Legendre function $P_\ell(\omega)$ when $m \neq 0$:

$$P_{\ell m}(\omega) = (1-\omega^2)^{|m|/2} \frac{d^{|m|}}{d\omega^{|m|}} P_\ell(\omega) \qquad (32)$$

The constant $N_{\ell m}$ can be defined by the general expression of $P_{\ell m}$:

$$N_{\ell m} = \sqrt{\frac{2\ell+1}{2} \frac{(\ell-|m|)!}{(\ell+|m|)!}} \qquad (33)$$

For $\ell = 0, 1, 2$, it is not as difficult to prove the given equations (25)~(26') satisfies the equations (31)~(33). In order to show this in generalization, we can apply the Leibniz rule over and over. (see. Tetsuro Inui "Tokushu-kansu" Iwanami zensho)

To put in order, we can say that the eigenfunction of orbital angular momentum $\hat{\ell}^2$ and $\hat{\ell}_z$ are:

$$Y_{\ell m}(\theta, \phi) = \Theta_{\ell m}(\theta)\Phi_m(\phi), \qquad (34a)$$

$$\Theta_{\ell m}(\theta) = (-1)^{\frac{m+|m|}{2}} [\frac{2\ell+1}{2} \frac{(\ell-|m|)!}{(\ell+|m|)!}]^{1/2} P_{\ell m}(\cos\theta) \qquad (34b)$$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \qquad (34c)$$

$Y_{\ell m}(\theta, \phi)$ is called spherical function or spherical surface harmonics, and the operation of $\hat{\ell}^2, \hat{\ell}_z$ can give the results:

$$\hat{\ell}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m} \qquad (35a)$$

$$\hat{\ell}_z Y_{\ell m} = \hbar m Y_{\ell m} \qquad (35b)$$

The factor $(-1)^{\frac{m+|m|}{2}}$ in (34b) is usually taken, and the reason why we need this particular factor will be found in later on, but for now, think of it as an idiomatic expression. In (24)~(25'), what is written in terms of $\varphi_{\ell m}$ can be expressed in spherical function:

$$\varphi_{\ell m}(r, \theta, \phi) = r^\ell Y_{\ell m}(\theta, \phi)$$

Spherical function $Y_{\ell m}(\theta,\phi)$ is considered as the eigenfunction, which includes Hermitian operator $\hat{\ell}^2$ and $\hat{\ell}_z$ with the eigenvalue $\hbar^2\ell(\ell+1)$ and $\hbar m$. Commonly, it is known that the eigenfunctions of Hermitian operators with different eigenvalues are orthogonal to one another; hence there is an orthonormality relationship to be established. It is of course possible to make a proof by using the specific formations of the spherical function, though it may take a tremendous time and effort to complete.

Spatial Images of Spherical Function

To have an ability to picture the images of the wavefunctions can be a very essential matter henceforward. To provide an assistance to improve the skill, we will show the behaviors of the spherical functions in a space (Fig.6.2). In the figure, also seen in (24), (25), (26), the value ℓ represents the number of the nodal surface for the wavefunction on the spherical surface. (Momentum/$\hbar$) corresponds to the number of nodes for the unit length of the plane waves, in the same way, (angular momentum/$\hbar$) corresponds to the number of nodal surface for the oscillating body on spherical surface. The value $|m|$ characterizes the form of the wavefunction when rotated about the z-axis. Therefore, the parity of ℓ:$(-1)^\ell$ may give a sign changing in wavefunction toward a space inversion $r \to -r$. Although, there is no reason for Hamiltonian and a space to have a special attention to z-axis, thus it may seem quite strange for the wavefunction and $\hat{\ell}_z$ of their eigenvalues to hold a special meaning; it is simply a matter of how the base is selected but nothing else.

-----Fig.6-2------

Step-up Operator and Step-down Operator (abbrev. in lecture)

Let's take a closer look at the associated Legendre differential equations $P_{\ell m}(\omega)$. We define $m \geq 0$, and differentiate (32) in terms of ω then multiply by $\sqrt{1-\omega^2}$ to obtain:

$$\sqrt{1-\omega^2}\frac{d}{d\omega}P_{\ell m} = -\frac{m\omega}{\sqrt{1-\omega^2}}P_{\ell m} + P_{\ell m+1} \qquad (37a)$$

We can reform the associated Legendre differential equations (29'):

$$\{\sqrt{1-\omega^2}\frac{d}{d\omega} - \frac{(m+1)\omega}{\sqrt{1-\omega^2}}\}\{\sqrt{1-\omega^2}\frac{d}{d\omega} + \frac{m\omega}{\sqrt{1-\omega^2}}\}P_{\ell m} + (\ell(\ell+1) - m(m+1))P_{\ell m} = 0$$

Take (37a) for the first clause, and define $m+1 \to m$:

$$\sqrt{1-\omega^2}\frac{d}{d\omega}P_{\ell m} = \frac{m\omega}{\sqrt{1-\omega^2}}P_{\ell m} - (\ell+m)(\ell-m+1)P_{\ell m-1} \qquad (37b)$$

Keep in mind that $m \geq 1$ in (37b). Reconstitute the variable from ω to θ, by $\omega = \cos\theta$:

$$\frac{d}{d\theta} = -\sin\theta\frac{d}{d(\cos\theta)} = -\sqrt{1-\omega^2}\frac{d}{d\omega}$$

Each equation (37a~b) then, be reformed as:

$$m \geq 0: \quad \frac{dP_{\ell m}}{d\theta} = m\cot\theta P_{\ell m} - P_{\ell m+1}, \qquad (37a')$$

$$m \geq 1: \quad \frac{dP_{\ell m}}{d\theta} = -m\cot\theta P_{\ell m} + (\ell+m)(\ell-m+1)P_{\ell m-1} \qquad (37b')$$

In the next step, rewrite the equation of $P_{\ell m}$ by the terms of $\Theta_{\ell m}$ using (34b):

$$m \geq 0: \quad \frac{d\Theta_{\ell m}}{d\theta} = m\cot\theta\Theta_{\ell m} + \sqrt{(\ell-m)(\ell+m+1)}\Theta_{\ell m+1},$$
$$m \geq 1: \quad \frac{d\Theta_{\ell m}}{d\theta} = -m\cot\theta\Theta_{\ell m} - \sqrt{(\ell+m)(\ell-m+1)}\Theta_{\ell m-1}$$

Given $\Theta_{\ell 1} = -\Theta_{\ell -1}$, we can obtain the second equation by defining $m = 0$ for the first equation. In this way, we can consider the limit of the second equation $m \geq 1$ as $m \geq 0$. Now, we substitute $\Theta_{\ell|m|} = (-1)^m\Theta_{\ell-|m|}$ into the first equation to obtain:

$$\frac{d\Theta_{\ell-|m|}}{d\theta} = |m|\cot\theta\Theta_{\ell-|m|} + (-1)^m\sqrt{(\ell-|m|)(\ell+|m|+1)}\Theta_{\ell|m|+1}$$

Where it is $m < 0$, then $|m| = -m,\ |m|+1 = -m+1 = -(m-1),$

$$\Theta_{\ell|m|+1} = \Theta_{\ell-(m-1)} = (-1)^{m-1}\Theta_{\ell m-1}$$

Therefore, the first equation can be finally reformed as:

$$\frac{d\Theta_{\ell m}}{d\theta} = -m\cot\theta\Theta_{\ell m} - \sqrt{(\ell+m)(\ell-m+1)}\Theta_{\ell m-1}, \quad (m<0)$$

Which indicates that the second equation is valid under the condition of $m < 0$, and in the same way, through the second equation, we can derive the validity of the first equation under the condition of $m < 0$. In general, without any concerns over the sings of m , the recurrence formula can be determined:

$$\begin{aligned}\frac{d\Theta_{\ell m}(\theta)}{d\theta} &= +m\cot\theta\Theta_{\ell m}(\theta) + \sqrt{(\ell-m)(\ell+m+1)}\Theta_{\ell m+1}(\theta) \\ &= -m\cot\theta\Theta_{\ell m}(\theta) - \sqrt{(\ell+m)(\ell-m+1)}\Theta_{\ell m-1}(\theta)\end{aligned} \qquad (38)$$

We define new angular momentum operators $\hat{\ell}_+, \hat{\ell}_-$ as follows:

$$\begin{aligned}\hat{\ell}_+ &= \hat{\ell}_x + i\hat{\ell}_y = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), \\ \hat{\ell}_- &= \hat{\ell}_x - i\hat{\ell}_y = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right).\end{aligned} \qquad (39)$$

In considering $\frac{\partial}{\partial\phi}Y_{\ell m} = i\hbar mY_{\ell m}$, the two equations from (38) represent the operation of the spherical function to $\hat{\ell}_\pm$:

$$\hat{\ell}_\pm Y_{\ell m}(\theta,\phi) = \hbar\sqrt{(\ell \mp m)(\ell \pm m + 1)}Y_{\ell m\pm 1}(\theta,\phi) \qquad (40)$$

That is to say, the new operator $\hat{\ell}_\pm$ has an ability to change the eigenstate of the angular momentum (ℓ, m) to be $(\ell, m \pm 1)$, and the reason why we picked a certain sign for the equations (25)~(26') was to avoid the change in the signs of the equation at (40). $\hat{\ell}_+$, $\hat{\ell}_-$ are often called step-up operator and step-down operator respectively. The commutation relation of $\hat{\ell}_\pm$ can be determined with definition equation (39) and (10) to be written as:

$$[\hat{\ell}_+, \hat{\ell}_-] = 2\hbar\hat{\ell}_z, \qquad (41a)$$

$$[\hat{\ell}_\pm, \hat{\ell}_z] = \mp\hbar\hat{\ell}_\pm. \qquad (41b)$$

As an appendix, we write out the equation of $\hat{\ell}^2$ reformed in terms of $\hat{\ell}_\pm$ as well as in $\hat{\ell}_z$:

$$\hat{\ell}^2 = \frac{1}{2}(\hat{\ell}_+\hat{\ell}_- + \hat{\ell}_-\hat{\ell}_+) + \hat{\ell}_z^2 = \hat{\ell}_+\hat{\ell}_- + \hat{\ell}_z^2 - \hbar\hat{\ell}_z = \hat{\ell}_-\hat{\ell}_+ + \hat{\ell}_z^2 + \hbar\hat{\ell}_z. \; (41c)$$