------Quantum Mechanics II: Angular Momentum----

Eigenfunction of Orbital Angular Momentum, e.g. Cubic Harmonics

$(\ell = 0 : s - wave) \quad \varphi_{00}(x, y, z) = \frac{1}{\sqrt{4\pi}}$

$(\ell = 1 : p - wave)\ \varphi_{1x} = \frac{\sqrt{3}}{2\sqrt{\pi}}x, \quad \varphi_{1y} = \frac{\sqrt{3}}{2\sqrt{\pi}}y, \quad \varphi_{1z} = \frac{\sqrt{3}}{2\sqrt{\pi}}z$

$$\varphi_{11} = -\frac{1}{\sqrt{2}}(\varphi_{1x} + i\varphi_y) = -\sqrt{\frac{3}{8\pi}}(x + iy) = -\sqrt{\frac{3}{8\pi}}r\sin\theta e^{i\phi},$$

$$\varphi_{10} = \varphi_{1z} = \sqrt{\frac{3}{4\pi}}z = \sqrt{\frac{3}{4\pi}}r\cos\theta,$$

$$\varphi_{1-1} = \frac{1}{\sqrt{2}}(\varphi_{1x} - i\varphi_{1y}) = \sqrt{\frac{3}{8\pi}}(x - iy) = \sqrt{\frac{3}{8\pi}}r\sin\theta e^{-i\phi}$$

$(\ell = 2 : d - wave)$

$$\varphi_{2,3z^2-r^2} = \sqrt{\frac{5}{16\pi}}(3z^2 - r^2) = \sqrt{\frac{5}{16\pi}}r^2(3\cos^2\theta - 1),$$

$$\varphi_{2,x^2-y^2} = \sqrt{\frac{15}{16\pi}}(x^2 - y^2) = \sqrt{\frac{15}{16\pi}}r^2\sin^2\theta\cos 2\phi,$$

$$\varphi_{2,yz} = \sqrt{\frac{15}{4\pi}}yz = \sqrt{\frac{15}{4\pi}}r^2\sin\theta\cos\theta\sin\phi,$$

$$\varphi_{2,zx} = \sqrt{\frac{15}{4\pi}}zx = \sqrt{\frac{15}{4\pi}}r^2\sin\theta\cos\theta\cos\phi,$$

$$\varphi_{2,xy} = \sqrt{\frac{15}{4\pi}}xy = \sqrt{\frac{15}{4\pi}}r^2\sin^2\theta\sin 2\phi.$$

$$\varphi_{22} = \frac{1}{\sqrt{2}}(\varphi_{2,x^2-y^2} + i\varphi_{2,xy}) = \frac{\sqrt{15}}{4\sqrt{2\pi}}r^2\sin^2\theta e^{2i\phi},$$

$$\varphi_{21} = -\frac{1}{\sqrt{2}}(\varphi_{2,zx} + i\varphi_{2,yz}) = -\frac{\sqrt{15}}{2\sqrt{2\pi}}r^2\sin\theta\cos\theta e^{i\phi},$$

$$\varphi_{20} = \varphi_{2,3z^2-r^2} = \frac{\sqrt{5}}{2\sqrt{4\pi}}r^2(3\cos^2\theta - 1),$$

$$\varphi_{2-1} = \frac{1}{\sqrt{2}}(\varphi_{2,zx} - i\varphi_{2,yz}) = \frac{\sqrt{15}}{2\sqrt{2\pi}}r^2\sin\theta\cos\theta e^{-i\phi},$$

$$\varphi_{2-2} = \frac{1}{\sqrt{2}}(\varphi_{2,x^2-y^2} - i\varphi_{2,xy}) = \frac{\sqrt{15}}{4\sqrt{2\pi}}r^2\sin^2\theta e^{-2i\phi}$$

Eigenfunction of Orbital Angular Momentum---Spherical Functions---

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP_{\ell m}(\cos\theta)}{d\theta}) + (\ell(\ell+1) - \frac{m^2}{\sin^2\theta})P_{\ell m}(\cos\theta) = 0.$$

Legendre Associated Differential Equations

$$\frac{d}{d\omega}[(1-\omega^2)\frac{dP_{\ell m}(\omega)}{d\omega}] + (\ell(\ell+1) - \frac{m^2}{1-\omega^2})P_{\ell m}(\omega) = 0$$

Legendre Differential Equations

$$\frac{d}{d\omega}[(1-\omega^2)\frac{dP_{\ell}(\omega)}{d\omega}] + \ell(\ell+1)P_{\ell}(\omega) = 0$$

$$P_{\ell}(\omega) = \frac{1}{2^{\ell}\ell!}\frac{d^{\ell}}{d\omega^{\ell}}(\omega^2-1)^{\ell} = \frac{(-1)^{\ell}}{2^{\ell}\ell!}\frac{d^{\ell}}{d\omega^{\ell}}(1-\omega^2)^{\ell}$$

$$P_{\ell m}(\omega) = (1-\omega^2)^{|m|/2}\frac{d^{|m|}}{d\omega^{|m|}}P_{\ell}(\omega)$$

$$N_{\ell m} = \sqrt{\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}}$$

Eigenfunction of Orbital Angular Momentum $\hat{\ell}^2$ and $\hat{\ell}_z$ (Spherical Functions)

$$Y_{\ell m}(\theta,\phi) = \Theta_{\ell m}(\theta)\Phi_m(\phi),$$

$$\Theta_{\ell m}(\theta) = (-1)^{\frac{m+|m|}{2}}[\frac{2\ell+1}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}]^{1/2}P_{\ell m}(\cos\theta)$$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

$$\hat{\ell}^2 Y_{\ell m} = \hbar^2\ell(\ell+1)Y_{\ell m}$$

$$\hat{\ell}_z Y_{\ell m} = \hbar m Y_{\ell m}$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y_{\ell' m'}(\theta,\phi)^* Y_{\ell m}(\theta,\phi) = \delta_{\ell'\ell}\delta_{m'm}$$

Matrix Element of Angular Momentum Operator as well as the Step-up and Step-down Operator

A specific expression of the orbital angular momentum operator:

$$\hat{\ell}_{\pm}Y_{\ell m}(\theta,\phi) = \hbar\sqrt{(\ell\mp m)(\ell\pm m+1)}Y_{\ell m\pm 1}(\theta,\phi)$$