Orbital Angular Momentum: Symmetry and Conservation Law

Conservation of Orbital Angular Momentum

Let's say, we have a particle in a central force field. Take an origin of the coordinate as a center of the force, and position the particle as $\boldsymbol{r}$, then the vectors of the force can be expressed as $f(r)\boldsymbol{r}/r$. Given a mass of the particle as m, we can rewrite Newton's Equation on Motion:

$$m\ddot{\boldsymbol{r}} = f(r)\frac{\boldsymbol{r}}{r} \tag{1}$$

The dots on the variable represent a time-derivative. The time-derivative for the vector product $\boldsymbol{r} \times \dot{\boldsymbol{r}}$ according to (1) can be expressed as:

$$\frac{d}{dt}(\boldsymbol{r} \times \dot{\boldsymbol{r}}) = (\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}}) + (\boldsymbol{r} \times \ddot{\boldsymbol{r}}) = \frac{f(r)}{m}\frac{(\boldsymbol{r} \times \boldsymbol{r})}{r} = 0$$

Therefore, $\boldsymbol{r} \times \dot{\boldsymbol{r}}$ remains constant independent of time. The vectors that we consider are the angular momentum (orbital angular momentum) in classical mechanics. ($\boldsymbol{p} = m\dot{\boldsymbol{r}}$ is the momentum)

$$\vec{\ell} = \boldsymbol{r} \times m\dot{\boldsymbol{r}} = \boldsymbol{r} \times \boldsymbol{p} \tag{2}$$

The equation above shows that the angular momentum is kept constant on the particle where it moves within a central force field. The constant direction of $\vec{\ell}$ represents the position vector $\boldsymbol{r}$ always stays on a one plane, which is perpendicular to $\vec{\ell}$. We can understand the meaning of a vector having constant magnitude, by simply following the steps: First, take $\vec{\ell}$ direction on z-axis and using two-dimension polar coordinates:

$$x = r\cos\phi, \quad y = r\sin\phi$$

Classic relational expression for $\ell_z = xp_y - yp_z$:

$$p_x = m\dot{r}\cos\phi - mr\sin\phi\dot{\phi}, \qquad p_y = m\dot{r}\sin\phi + mr\cos\phi\dot{\phi}$$

We can derive:

$$\ell_z = mr^2\frac{d\phi}{dt}$$

Accordingly, $\ell =$constant (now we define $\ell_z =$constant) refers to the constant, which is independent of areal velocity $r^2\frac{d\phi}{dt}$. We categorize angular momentum especially for the orbital to differentiate the spin angular momentum that is a purely quantum mechanic phenomenon without any analogy in classical mechanics.

The total angular momentum remains constant even when a system of particles interact with one another, and the vector of the force acting on in between the particles is parallel to the vector $r_{ik} = r_i - r_k$. The total angular momentum $\boldsymbol{L}$ is defined as the sum of the angular momentum of each particle with attachment i, which refers to the individual particle:

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times m\dot{\boldsymbol{r}}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i \tag{3}$$

Time derivative of (3) is:

$$\begin{aligned}\frac{d}{dt}\boldsymbol{L} &= \sum_i m\{(\dot{\boldsymbol{r}}_i \times \dot{\boldsymbol{r}}_i) + (\boldsymbol{r}_i \times \ddot{\boldsymbol{r}}_i)\} = \sum_i (\boldsymbol{r}_i \times \sum_k \boldsymbol{F}_{ik}) \\ &= \sum_{i>k} (\boldsymbol{r}_i - \boldsymbol{r}_k) \times \boldsymbol{F}_{ik} = 0\end{aligned} \tag{4}$$

Above indicates that the total angular momentum is being conserved. $\boldsymbol{F}_{ik}$ is the force acting on the particle i from the particle k, and the equation of motion can be written with applications of Newton's third law $\boldsymbol{F}_{ik} = -\boldsymbol{F}_{ki}$, and the fact of $\boldsymbol{r}_i - \boldsymbol{r}_k$ being parallel to $\boldsymbol{F}_{ik}$:

$$m\ddot{\boldsymbol{r}}_i = \sum_k \boldsymbol{F}_{ik}$$

In classical mechanics, the angular momentum is conserved when the particle moves in a central force field or when the particles are interacting with a force acting along a direction of the mutual position vectors,.

Orbital Angular Momentum Operator in Quantum Mechanics

The orbital angular momentum is defined by (2), hence we can rewrite (2) as an operator:

$$\begin{aligned}\hat{\ell} = \boldsymbol{r} \times \hat{\mathbf{p}} = \frac{\hbar}{i}\boldsymbol{r} \times \nabla &= \frac{\hbar}{i}(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \\ &= (\hat{\ell}_x, \hat{\ell}_y, \hat{\ell}_z)\end{aligned} \tag{5}$$

The component $\hat{\ell}_x$, when acted about the function $f(r)$, we can obtain:

$$\frac{\partial f(r)}{\partial x} = \frac{df(r)}{dr} \cdot \frac{\partial r}{\partial x} = \frac{x}{r} \cdot \frac{df(r)}{dr}$$

We can derive:

$$\hat{\ell}_x f(r) = -i\hbar(y\frac{\partial f(r)}{\partial z} - z\frac{\partial f(r)}{\partial y}) + f(r)\hat{\ell}_x = f(r)\hat{\ell}_x \tag{6a}$$

In the same way, we can also derive:

$$\hat{\ell}_y f(r) = f(r)\hat{\ell}_y, \quad \hat{\ell}_z f(r) = f(r)\hat{\ell}_z \tag{6b}$$

As we can see later on in (16), the above refers to the fact that $\hat{\ell}_x$ and others does not include the derivative $\partial/\partial r$ of the radial vector r in polar coordinates.

Now, take a look at the close relationship between the operators of the orbital angular momentum and Laplacian $\Delta = \nabla \cdot \nabla$. Apparently we can write out:

$$\begin{aligned} \hat{\ell}_x^2 &= (\frac{\hbar}{i})^2 (y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}) \\ &= -\hbar^2 (y^2\frac{\partial^2}{\partial z^2} + z^2\frac{\partial^2}{\partial y^2} - 2yz\frac{\partial^2}{\partial y \partial z} - y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}) \end{aligned}$$

Then,

$$\begin{aligned} &(\hat{\ell}_x^2 + \hat{\ell}_y^2 + \hat{\ell}_z^2)/\hbar^2 \\ &= -(y^2\frac{\partial^2}{\partial z^2} + z^2\frac{\partial^2}{\partial y^2} + z^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial z^2} + x^2\frac{\partial^2}{\partial y^2} + y^2\frac{\partial^2}{\partial x^2}) \\ &\quad + 2(yz\frac{\partial^2}{\partial y \partial z} + zx\frac{\partial^2}{\partial z \partial x} + xy\frac{\partial^2}{\partial x \partial y}) + 2(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + x\frac{\partial}{\partial x}) \end{aligned} \tag{7a}$$

While it is:

$$\begin{aligned} r^2\Delta =& (x^2 + y^2 + z^2)(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) \\ =& (x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + z^2\frac{\partial^2}{\partial z^2}) \\ &+ (y^2\frac{\partial^2}{\partial z^2} + z^2\frac{\partial^2}{\partial y^2} + z^2\frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial z^2} + x^2\frac{\partial^2}{\partial y^2} + y^2\frac{\partial^2}{\partial x^2}), \end{aligned} \tag{7b}$$

$$\begin{aligned} (\boldsymbol{r}\cdot\nabla)^2 =& (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})^2 \\ =& 2(xy\frac{\partial^2}{\partial x \partial y} + yz\frac{\partial^2}{\partial y \partial z} + zx\frac{\partial^2}{\partial z \partial x}) \\ &+ (x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + z^2\frac{\partial^2}{\partial z^2}) + (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}) \end{aligned} \tag{7c}$$

In short, from $(7a \sim c)$ we can derive:

$$\hat{\ell}^2/\hbar^2 = -r^2\Delta + (\boldsymbol{r}\cdot\nabla)^2 + (\boldsymbol{r}\cdot\nabla) \tag{8}$$

This is equally formulated. We can also rewrite the Laplacian:

$$\Delta = \frac{1}{r^2}[(\boldsymbol{r}\cdot\nabla)^2 + (\boldsymbol{r}\cdot\nabla)] - \frac{1}{r^2\hbar^2}\hat{\ell}^2 \qquad (9)$$

Furthermore, to express $(\boldsymbol{r}\cdot\nabla)$ in polar coordinates (note (15) and the graph 6.1) we can also write:

$$\Delta = \frac{1}{r^2}(r\frac{\partial}{\partial r})^2 + \frac{1}{r}(\frac{\partial}{\partial r}) - \frac{1}{r^2\hbar^2}\hat{\ell}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{1}{r^2\hbar^2}\hat{\ell}^2 \qquad (9')$$

As it is indicated in (6a~b), $\hat{\ell}$ does not include the derivatives of the radius vector component. We also obtain the fact that all the operators concerning with the angles θ, ϕ are included within the term $\hat{\ell}^2$ when Laplacian is in polar coordinates because we cannot observe θ, ϕ in any other terms but $\hat{\ell}^2$, as it shows in (9').

There is another essential relational expression to be derived. The commutator $[\hat{\ell}_\alpha, \hat{\ell}_\beta]$ is easily found out by using the definition (5):

$$\begin{aligned}
&[\hat{\ell}_x, \hat{\ell}_y] \\
&= (\frac{\hbar}{i})^2\{(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}) - (z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})\} \\
&= (\frac{\hbar}{i})^2\{y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\} = i\hbar\hat{\ell}_z.
\end{aligned} \qquad (10a)$$

In the same way, we can calculate the followings:

$$[\hat{\ell}_y, \hat{\ell}_z] = i\hbar\hat{\ell}_x, \quad [\hat{\ell}_z, \hat{\ell}_x] = i\hbar\hat{\ell}_y \qquad (10b)$$

$$[\hat{\ell}_x, \hat{\ell}_x] = [\hat{\ell}_y, \hat{\ell}_y] = [\hat{\ell}_z, \hat{\ell}_z] = 0 \qquad (10c)$$

For another expression we can write:

$$\begin{aligned}
[\hat{\ell}_x, \Delta] = \frac{\hbar}{i}\{&(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) \\
&- (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})\} \\
= \hat{\ell}_x\Delta - \hat{\ell}_x\Delta = 0.&
\end{aligned} \qquad (11a)$$

In exactly the same way, we can derive:

$$[\hat{\ell}_y, \Delta] = [\hat{\ell}_z, \Delta] = 0 \qquad (11b)$$

As we have already shown in above, there exist a relationship (9') between the Laplacian Δ and the sum of the square of the angular momentum operators. Moreover, $\hat{\ell}_\alpha$ does not include the derivative $\frac{\partial}{\partial r}$ of the radius vector component. Based on these facts, (11a~b) indicates:

$$[\hat{\ell}_x, \hat{\ell}^2] = [\hat{\ell}_y, \hat{\ell}^2] = [\hat{\ell}_z, \hat{\ell}^2] = 0 \qquad (11c)$$

(11c) can be derived through the direct calculation of the commutators $\hat{\ell}^2$ and $\hat{\ell}_x, \hat{\ell}_y, \hat{\ell}_z$, also.