

Mathematical Statistics

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<http://ishiken.free.fr/lecture.html>

Oct. 18 Combination and probability

Oct. 25 Random variables and probability
distributions

Nov. 1 Representative probability distributions

Nov. 8 (First half) Random walk and gambler's
ruin problem

(Latter half) Brownian motion and diffusion

} Stochastic process

Nov. 22 Noise theory

Only exercises are provided on Nov.
15.

Reference books

- [1] Satsuma, J. (2001). “*Probability /Statistics*” -- *Beginning course of mathematics of science and technology 7*, Iwanami Shoten
- [2] Kolmogorov, A.N., Žurbenko, I. G., Prokhorov, A.V. (2003). “*Beginning of Kolmogorov’s Probability Theory*”. (Trans. Murayama , T. & Baba, Y.): Morikita Shuppan
- [3] Kitahara, K. (1997). “*Nonequilibrium Statistical Mechanics*” – *Iwanami Fundamental Physics Series 8*, Iwanami Shoten

Mathematical Statistics

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Nov. 22 Noise theory

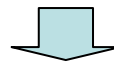
- Wiener-Khintchine's theorem
- Nyquist's theorem

6-1 Wiener-Khintchine's theorem

→ Reference book [3] p.103

- Noises generated in electric circuits
 - Example of phenomena to be represented via models including random force
- ↓
- Voltage $V(t)$ generated at the both ends of an electrical resistive element
 - $V(t)=0$ is ideal
 - In reality:

Thermal noises of conduction electrons are generated inside.



$V(t)$ fluctuates around zero (noises).

Consider what frequency elements are contained within such noises. → **Power Spectrum**

6-1 Wiener-Khintchine's Theorem

- Power Spectrum

- Noise $V(t)$ to be observed for a long stretch of time T

Minimum unit of frequency $f \frac{1}{T} \Rightarrow f_n = \frac{n}{T} \quad (n = \pm 1, \pm 2, \dots)$

$V(t)$ can be decomposed into the sums of these frequency elements.

$$V(t) = \sum_n e^{i2\pi f_n t} V_n \qquad V_n = \frac{1}{T} \int_0^T e^{-i2\pi f_n t} V(t) dt$$

Fourier transform

- Complex number in general $V_{-n} = V_n^*$

$V(t)$ Random $\Rightarrow V_n$ Random

Consider the average of $|V_n|^2$

Absolute value square of amplitude \rightarrow Intensity of each vibration element

6-1 Wiener-Khintchine's Theorem

- Power Spectrum → Reference book [3], p. 104

Consider the average of $|V_n|^2$.

↙
Absolute value square of amplitude → Intensity of each vibration element

Intensity of amplitude included in minute width of frequency Δf

↙ Average of noise samples

Power Spectrum

$$S_V(f)\Delta f = 2 \sum_{f < f_n < f + \Delta f} \langle |V_n|^2 \rangle$$

$f_n = \frac{n}{T} \Rightarrow$ The number of frequencies included in the width of Δf can be obtained from the formulae shown in the right: $\Delta f \div \left(\frac{1}{T}\right) = T\Delta f$

$$S_V(f)\Delta f = 2 \sum_{f < f_n < f + \Delta f} \langle |V_n|^2 \rangle \Rightarrow S_V(f)\Delta f = 2T \langle |\hat{V}(f)|^2 \rangle \Delta f \Rightarrow S_V(f) = 2T \langle |\hat{V}(f)|^2 \rangle$$

$$\hat{V}(f) = \frac{1}{T} \int_0^T e^{-i2\pi ft} V(t) dt$$

6-1 Wiener-Khintchine's Theorem

• Power Spectrum

$$S_V(f) = 2T \left\langle \left| \hat{V}(f) \right|^2 \right\rangle \quad \hat{V}(f) = \frac{1}{T} \int_0^T e^{-i2\pi ft} V(t) dt$$

$$S_V(f) = \frac{2}{T} \int_0^T dt_1 \int_0^T e^{-i2\pi f(t_1-t_2)} \langle V(t_1)V^*(t_2) \rangle dt_2$$

Steady State (equilibrium state)

Time correlation function of noise

$$\langle V(t_1)V^*(t_2) \rangle = \phi_V(t_1 - t_2)$$

$$\begin{aligned} S_V(f) &= \frac{2}{T} \int_0^T dt_1 \int_0^T e^{-i2\pi f(t_1-t_2)} \phi_V(t_1 - t_2) dt_2 \\ &= \frac{2}{T} \int_0^T dt_1 \int_0^{t_1} \left[e^{-i2\pi f(t_1-t_2)} \phi_V(t_1 - t_2) + e^{-i2\pi f(t_2-t_1)} \phi_V(t_2 - t_1) \right] dt_2 \\ &= \frac{2}{T} \int_0^T dt_1 \int_0^{t_1} e^{-i2\pi f(t_1-t_2)} \phi_V(t_1 - t_2) dt_2 + \text{c.c.} \\ &= 2 \int_0^T \left(1 - \frac{t}{T}\right) e^{-i2\pi ft} \phi_V(t) dt + \text{c.c.} = 4 \int_0^T \left(1 - \frac{t}{T}\right) \text{Re} \left[e^{-i2\pi ft} \phi_V(t) \right] dt \end{aligned}$$

6-1 Wiener-Khintchine's Theorem

- Power Spectrum

$$S_V(f) = 4 \int_0^T \left(1 - \frac{t}{T}\right) \operatorname{Re} \left[e^{-i2\pi ft} \phi_V(t) \right] dt \rightarrow 4 \int_0^\infty \operatorname{Re} \left[e^{-i2\pi ft} \phi_V(t) \right] dt$$

$\phi_V(t)$ is a damping function.

Wiener-Khintchine's Theorem

- Power Spectrum is represented as the integral of time correlation functions of noises (Fourier transform).

$$S_V(f) = 4 \int_0^\infty \operatorname{Re} \left[e^{-i2\pi ft} \phi_V(t) \right] dt$$

→ Reference book [3],
p. 105

- White Noise

$\phi(t) = 2D_V \delta(t)$ Noises generated at different times are not correlated at all with each other.

$$S_V(f) = 4D_V \quad \text{Constant independent of frequency} \quad \Rightarrow \quad \text{White Noise}$$

6-2 Nyquist's Theorem

6-2 Nyquist's Theorem

When power spectral intensity of the fluctuation in thermal noises, generated at the both ends of a resistor having resistance value R , is $S_V(f) = 4 D_V$ (White Noise),

$$D_V = Rk_B T$$

→ Reference book [3], p.106

RC Circuit

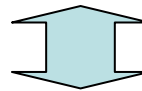
$$\frac{dQ}{dt} = -\frac{Q}{CR} + \frac{V(t)}{R}$$

Ohm's law

Electromotive force caused by thermal noises

Brownian motion

$$\frac{du}{dt} = -\gamma u + \frac{R(t)}{m}$$



$$\begin{aligned} Q &\Leftrightarrow u \\ \frac{1}{CR} &\Leftrightarrow \gamma \end{aligned}$$

6-2 Nyquist's Theorem

- Nyquist's Theorem

$$\frac{dQ}{dt} = -\frac{Q}{CR} + \frac{V(t)}{R}$$

$$Q(t) = Q(0)\exp\left(-\frac{t}{CR}\right) + \int_0^t \frac{V(t')}{R} \exp\left(-\frac{t-t'}{CR}\right) dt'$$

$$\begin{aligned} \langle [Q(t)]^2 \rangle_{\text{eq}} &= \langle [Q(0)]^2 \rangle_{\text{eq}} \exp\left(-\frac{2t}{CR}\right) + \left\langle \left[\int_0^t \frac{V(t')}{R} \exp\left(-\frac{t-t'}{CR}\right) dt' \right]^2 \right\rangle_{\text{eq}} \\ &\quad + \exp\left(-\frac{t}{CR}\right) \int_0^t \frac{\langle Q(0)V(t') \rangle_{\text{eq}}}{R} \exp\left(-\frac{t-t'}{CR}\right) dt' \end{aligned}$$

Zero

$$\begin{aligned} \left\langle \left[\int_0^t \frac{V(t')}{R} \exp\left(-\frac{t-t'}{CR}\right) dt' \right]^2 \right\rangle_{\text{eq}} &= \frac{1}{R^2} \int_0^t dt_1 \int_0^t dt_2 \exp\left(-\frac{t-t_1}{CR}\right) \exp\left(-\frac{t-t_2}{CR}\right) \langle V(t_1)V(t_2) \rangle_{\text{eq}} \\ &= \frac{2D_V}{R^2} \int_0^t dt_1 \exp\left(-\frac{2(t-t_1)}{CR}\right) = \frac{CD_V}{R} \left[1 - \exp\left(-\frac{2t}{CR}\right) \right] \end{aligned}$$

$\phi(t_1 - t_2) = 2D_V \delta(t_1 - t_2)$

6-2 Nyquist's Theorem

- Nyquist's Theorem → Reference book [3] p.107

$$\langle [Q(t)]^2 \rangle_{\text{eq}} = \langle [Q(0)]^2 \rangle_{\text{eq}} \exp\left(-\frac{2t}{CR}\right) + \frac{CD_V}{R} \left[1 - \exp\left(-\frac{2t}{CR}\right)\right]$$

Equal at equilibrium →

$$\langle [Q(t)]^2 \rangle_{\text{eq}} = \frac{CD_V}{R}$$

Energy generated in a condenser due to electric charge $E = \frac{Q^2}{2C}$

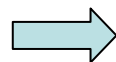
→ The realization probability of such fluctuation follows the Boltzmann distribution.

$$P_{\text{eq}}(Q) \propto \exp\left(-\frac{E}{k_B T}\right) = \exp\left(-\frac{Q^2}{2k_B TC}\right)$$



$$\langle Q^2 \rangle_{\text{eq}} = Ck_B T$$

$$\frac{CD_V}{R} = Ck_B T$$



$$D_V = Rk_B T$$



$$S_V(f) = 4Rk_B T$$

Nyquist's Theorem

Thermal noises are fixed through observations of macroscopic quantum.

6-2 Nyquist's Theorem

- **White Noise** → Reference book [3], p.107

$\phi(t) = 2D_V \delta(t)$ Noises generated at different times are not correlated at all with each other.

$S_V(f) = 4D_V$ Constant independent of frequency

} A sort of idealization

- **Lorentzian Noise** → Reference book [3], p.107

$\phi(t) = \langle V^2 \rangle e^{-t/\tau}$ Finite time constant τ in time correlation function of fluctuation

$S_V(f) = \langle V^2 \rangle \frac{4\tau}{(2\pi f\tau)^2 + 1}$ **Lorentzian noise**