Mathematical Statistics

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http://ishiken.free.fr/lecture.html

Oct. 18 Combination and probability
Oct. 25 Random variables and probability distributions
Nov. 1 Representative probability distributions
Nov. 8 (First half) Random walk and gambler's ruin problem
(Latter half) Brownian motion and diffusion

Stochastic process (Nov. 1 – Nov. 8)

Nov. 22 Noise theory

Only exercises are provided on Nov. 15.

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Reference books

[1] Satsuma, J. (2001). *"Probability /Statistics"* -- *Beginning course of mathematics of science and technology 7*, Iwanami Shoten

[2] Kolmogorov, A.N., Žurbenko, I. G., Prokhorov, A.V. (2003). *"Beginning of Kolmogorov's Probability Theory".* (Trans. Murayama , T. & Baba, Y.): Morikita Shuppan

[3] Kitahara, K. (1997). *"Nonequilibrium Statistical Mechanics" – Iwanami Fundamental Physics Series 8*, Iwanami Shoten

Mathematical Statistics

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Nov. 22 Noise theory

- Wiener-Khintchine's theorem
- Nyquist’s theorem

6-1 Wiener-Khintchine's theorem

→ Reference book [3] p.103

- Noises generated in electric circuits
  - Example of phenomena to be represented via models including random force

  ↓
- Voltage V(t) generated at the both ends of an electrical resistive element
  - V(t)=0 is ideal
  - In reality:

Thermal noises of conduction electrons are generated inside.

↓

V(t) fluctuates around zero (noises).

Consider what frequency elements are contained within such noises. → Power Spectrum

6-1 Wiener-Khintchine's Theorem

- Power Spectrum
  - Noise V(t) to be observed for a long stretch of time T

Minimum unit of frequency $f \frac{1}{T} \Rightarrow f_n = \frac{n}{T} \quad (n = \pm 1, \pm 2, \mathcal{L})$

$V(t)$ can be decomposed into the sums of these frequency elements.

$$V(t) = \sum_n e^{i2\pi f_n t} V_n \qquad V_n = \frac{1}{T}\int_0^T e^{-i2\pi f_n t} V(t)dt$$

Fourier transform

- Complex number in general $V_{-n} = V_n^*$

$V(t)$ Random $\Rightarrow$ V_n Random

Consider the average of $|V_n|^2$

Absolute value square of amplitude → Intensity of each vibration element

6-1 Wiener-Khintchine's Theorem

- Power Spectrum → Reference book [3], p. 104

Consider the average of $|V_n|^2$.

Absolute value square of amplitude → Intensity of each vibration element

Intensity of amplitude included in minute width of frequency Δf

Average of noise samples

Power Spectrum $S_V(f)\Delta f = 2 \sum_{f<f_n<f+\Delta f} \left\langle |V_n|^2 \right\rangle$

$f_n = \frac{n}{T}$ ⇨ The number of frequencies included in the width of Δf can be obtained from the formulae shown in the right: $\Delta f \div \left(\frac{1}{T}\right) = T\Delta f$

$$S_V(f)\Delta f = 2 \sum_{f<f_n<f+\Delta f} \left\langle |V_n|^2 \right\rangle \Rightarrow S_V(f)\Delta f = 2T\left\langle |\hat{V}(f)|^2 \right\rangle \Delta f \Rightarrow S_V(f) = 2T\left\langle |\hat{V}(f)|^2 \right\rangle$$

$$\hat{V}(f) = \frac{1}{T}\int_0^T e^{-i2\pi ft} V(t)dt$$

6-1 Wiener-Khintchine's Theorem

- Power Spectrum

$$S_V(f) = 2T\left\langle \left|\hat{V}(f)\right|^2 \right\rangle \qquad \hat{V}(f) = \frac{1}{T}\int_0^T e^{-i2\pi ft} V(t)dt$$

$$S_V(f) = \frac{2}{T}\int_0^T dt_1 \int_0^T e^{-i2\pi f(t_1-t_2)} \left\langle V(t_1)V^*(t_2) \right\rangle dt_2$$

Time correlation function of noise

Steady State (equilibrium state)

$$\left\langle V(t_1)V^*(t_2) \right\rangle = \phi_V(t_1 - t_2)$$

$$S_V(f) = \frac{2}{T}\int_0^T dt_1 \int_0^T e^{-i2\pi f(t_1-t_2)} \phi_V(t_1 - t_2) dt_2$$

$$= \frac{2}{T}\int_0^T dt_1 \int_0^{t_1} \left[e^{-i2\pi f(t_1-t_2)} \phi_V(t_1 - t_2) + e^{-i2\pi f(t_2-t_1)} \phi_V(t_2 - t_1) \right] dt_2$$

$$= \frac{2}{T}\int_0^T dt_1 \int_0^{t_1} e^{-i2\pi f(t_1-t_2)} \phi_V(t_1 - t_2) dt_2 + \text{c.c.}$$

$$= 2\int_0^T \left(1 - \frac{t}{T}\right) e^{-i2\pi ft} \phi_V(t) dt + \text{c.c.} = 4\int_0^T \left(1 - \frac{t}{T}\right) \mathrm{Re}\left[e^{-i2\pi ft} \phi_V(t) \right] dt$$

6-1 Wiener-Khintchine's Theorem

- Power Spectrum

$$S_V(f) = 4\int_0^T \left(1 - \frac{t}{T}\right) \mathrm{Re}\left[e^{-i2\pi ft}\phi_V(t)\right]dt \to 4\int_0^\infty \mathrm{Re}\left[e^{-i2\pi ft}\phi_V(t)\right]dt$$

$\phi_V(t)$ is a damping function.

Wiener-Khintchine's Theorem

- Power Spectrum is represented as the integral of time correlation functions of noises (Fourier transform).

$$S_V(f) = 4\int_0^\infty \mathrm{Re}\left[e^{-i2\pi ft}\phi_V(t)\right]dt$$

→ Reference book [3], p. 105

- White Noise

$\phi(t) = 2D_V\delta(t)$ Noises generated at different times are not correlated at all with each other.

$S_V(f) = 4D_V$ Constant independent of frequency ⇨ White Noise

6-2 Nyquist's Theorem

When power spectral intensity of the fluctuation in thermal noises, generated at the both ends of a resistor having resistance value R , is $S_V(f) = 4D_V$ (White Noise),

$$D_V = Rk_BT$$

→ Reference book [3], p.106

RC Circuit

$$\frac{dQ}{dt} = -\frac{Q}{CR} + \frac{V(t)}{R}$$

Ohm's law

Electromotive force caused by thermal noises

Brownian motion

$$\frac{du}{dt} = -\gamma u + \frac{R(t)}{m}$$

$Q \Leftrightarrow u$

$\frac{1}{CR} \Leftrightarrow \gamma$

6-2 Nyquist's Theorem

- Nyquist's Theorem

$$\frac{dQ}{dt} = -\frac{Q}{CR} + \frac{V(t)}{R}$$

$$Q(t) = Q(0)\exp\left(-\frac{t}{CR}\right) + \int_0^t \frac{V(t')}{R}\exp\left(-\frac{t-t'}{CR}\right)dt'$$

$$\left\langle [Q(t)]^2 \right\rangle_{\mathrm{eq}} = \left\langle [Q(0)]^2 \right\rangle_{\mathrm{eq}} \exp\left(-\frac{2t}{CR}\right) + \left\langle \left[\int_0^t \frac{V(t')}{R}\exp\left(-\frac{t-t'}{CR}\right)dt'\right]^2 \right\rangle_{\mathrm{eq}}$$

$$+\exp\left(-\frac{t}{CR}\right)\int_0^t \frac{\langle Q(0)V(t')\rangle_{\mathrm{eq}}}{R}\exp\left(-\frac{t-t'}{CR}\right)dt'$$

Zero

$$\left\langle \left[\int_0^t \frac{V(t')}{R}\exp\left(-\frac{t-t'}{CR}\right)dt'\right]^2 \right\rangle_{\mathrm{eq}} = \frac{1}{R^2}\int_0^t dt_1 \int_0^t dt_2 \exp\left(-\frac{t-t_1}{CR}\right)\exp\left(-\frac{t-t_2}{CR}\right)\langle V(t_1)V(t_2)\rangle_{\mathrm{eq}}$$

$$= \frac{2D_V}{R^2}\int_0^t dt_1 \exp\left(-\frac{2(t-t_1)}{CR}\right) = \frac{CD_V}{R}\left[1-\exp\left(-\frac{2t}{CR}\right)\right]$$

$$\phi(t_1 - t_2) = 2D_V\delta(t_1 - t_2)$$

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6-2 Nyquist's Theorem

- Nyquist's Theorem → Reference book [3] p.107

$$\left\langle [Q(t)]^2 \right\rangle_{eq} = \left\langle [Q(0)]^2 \right\rangle_{eq} \exp\left(-\frac{2t}{CR}\right) + \frac{CD_V}{R}\left[1 - \exp\left(-\frac{2t}{CR}\right)\right]$$

Equal at equilibrium ⇒

$$\left\langle [Q(t)]^2 \right\rangle_{eq} = \frac{CD_V}{R}$$

Energy generated in a condenser due to electric charge $E = \frac{Q^2}{2C}$

⇒ The realization probability of such fluctuation follows the Boltzmann distribution.

$$P_{eq}(Q) \propto \exp\left(-\frac{E}{k_B T}\right) = \exp\left(-\frac{Q^2}{2k_B TC}\right) \Rightarrow \left\langle Q^2 \right\rangle_{eq} = Ck_B T$$

$$\frac{CD_V}{R} = Ck_B T \Rightarrow D_V = Rk_B T \Rightarrow S_V(f) = 4Rk_B T$$

Nyquist's Theorem

Thermal noises are fixed through observations of macroscopic quantum.

6-2 Nyquist's Theorem

- White Noise → Reference book [3], p.107

$\phi(t) = 2D_V\delta(t)$ Noises generated at different times are not correlated at all with each other.

$S_V(f) = 4D_V$ Constant independent of frequency

} A sort of idealization

- Lorentzian Noise → Reference book [3], p.107

$\phi(t) = \langle V^2 \rangle e^{-t/\tau}$ Finite time constant τ in time correlation function of fluctuation

$$S_V(f) = \langle V^2 \rangle \frac{4\tau}{(2\pi f\tau)^2 + 1}$$

Lorentzian noise