Mathematical Statistics

Kenichi ISHIKAWA

http://ishiken.free.fr/lecture.html

Oct. 18 Combination and probability
Oct. 25 Random variables and probability distributions
Nov. 1 Representative probability distributions
Nov. 8 (First half) Random walk and gambler's ruin problem
(Latter half) Brownian motion and diffusion
Nov. 22 Noise theory

Stochastic process

Only exercises are provided on Nov. 15.

Reference books

[1] Satsuma, J. (2001). *“Probability /Statistics”* -- *Beginning course of mathematics of science and technology 7*, Iwanami Shoten

[2] Kolmogorov, A.N., Žurbenko, I. G., Prokhorov, A.V. (2003). *“Beginning of Kolmogorov’s Probability Theory”*. (Trans. Murayama , T. & Baba, Y.): Morikita Shuppan

[3] Kitahara, K. (1997). *“Nonequilibrium Statistical Mechanics”* – *Iwanami Fundamental Physics Series 8*, Iwanami Shoten

Mathematical Statistics

Kenichi ISHIKAWA

Nov. 8 (First half) Random walk and gambler's ruin problem

- Stochastic processes
- One-dimensional random walk
- Random walk and diffusion
- Gambler's ruin problem

4-1 Stochastic Processes

→ Reference book [1], p.176

Stochastic Process

Stochastic phenomena represented as a random variable $X(t)$ that changes with time.

Dynamic processes represented as a random variable that changes with time

- Random Walk
- Location of fine particles performing an erratic movement (eg, Brownian motion)
- Noises
- Population change in a city
- Ever-changing stock quote/foreign exchange rate

Nonequilibrium statistical mechanics

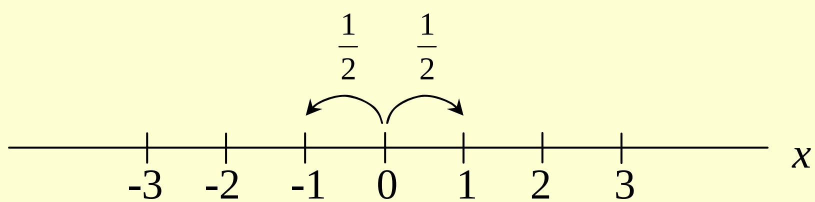
Mathematical finance/econophysics

Chaotic and unregulated movement of individual particles ⇔ Clear and simple regularity on the whole

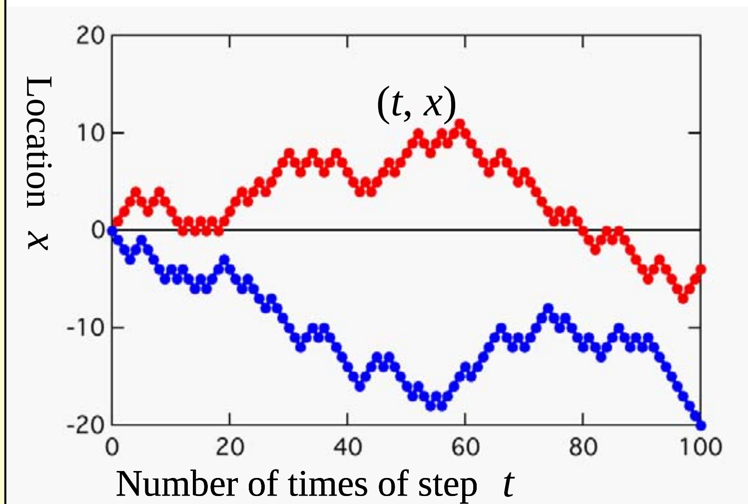
Theory of stochastic processes

4-2 One-Dimensional Random Walk

→ Reference book [2] , p.91



- A particle leaves x=0 at time t=0
- One-time jump allows one-space move to the right or left.
- Irrespective of location, the probability that the particle moves to the right or left is one-half.



$x(t)$: Location of a particle after t steps

$$x(0) = 0$$

$$x(1) = -1,\ 1$$

$$x(2) = -2,\ 0,\ 2$$

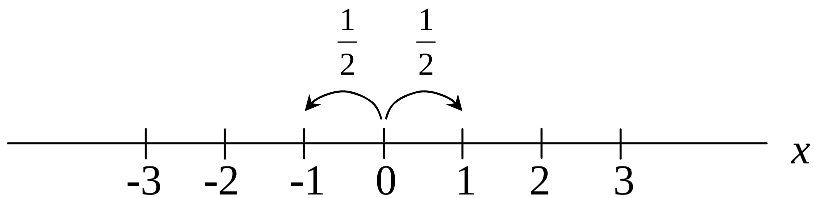
$$x(3) = -3,\ -1,\ 1,\ 3$$

$$x(t) = -t, -t+2, -t+4, \ldots, t-4, t-2, t = t-2k\ (k = 0, 1, \ldots, t)$$

$W(t,x)$ Probability to arrive at the location x after *t-times jumps*

$L(t,x)$ Number of possible routes arriving at the location x after t-times jumps

Number of routes arriving at (t, x) from zero



$$x(t) = -t, -t+2, -t+4, \cdots, t-4, t-2, t = t-2k \ (k = 0, 1, \cdots, t)$$

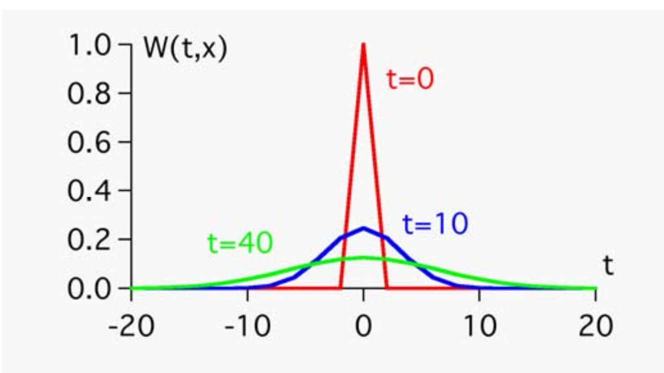
Binomial Distribution *Bin*(t,1/2)

$$W(t,x) = L(t,x)\left(\frac{1}{2}\right)^t = {}_tC_{(t-x)/2}\left(\frac{1}{2}\right)^t = {}_tC_{(t+x)/2}\left(\frac{1}{2}\right)^t$$

Jump to the right t - k times

Jump to the left k times

$$L(t,x) = {}_tC_k = {}_tC_{(t-x)/2} = {}_tC_{(t+x)/2}$$



Diffusion phenomenon of a kind

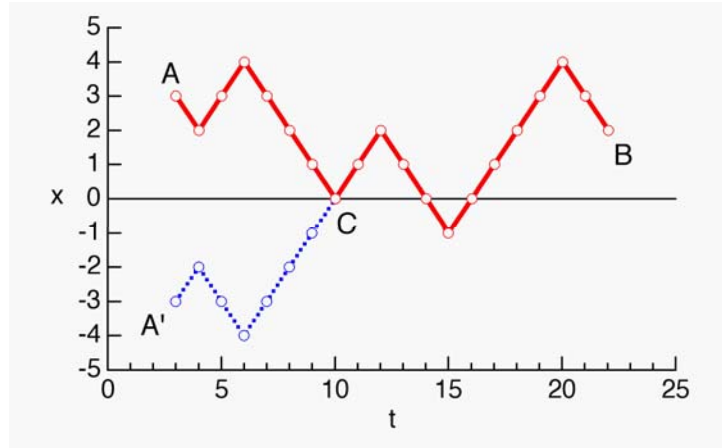
▪Reflection Principle

→ Reference book [2], p. 93

Symmetric with respect to the horizontal axis $A(t_0, x_0) \Longleftrightarrow A'(t_0, -x_0)$

$B(t, x)$

$0 \le t_0 < t \quad x_0 > 0 \quad x > 0$



> The number of routes that contact or intersect the horizontal axis from among those from A to B is equal to the total number of the routes from A' to B.

This Principle drastically facilitates the calculation of the number of routes meeting given requirements.

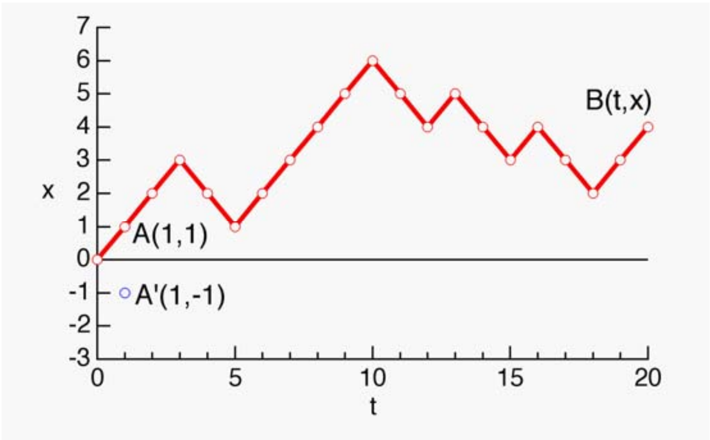
Positive passage: Every passing point stays above the horizontal axis throughout a passage at any time. $(x > 0)$

Non-negative passage: Every passing point does not lie below the horizontal axis throughout a passage. $(x \ge 0)$

Likewise, negative passage and non-positive passage can be defined.

[Example] How many positive passages from zero to the point $B(t,x), 0 < x \le t$?

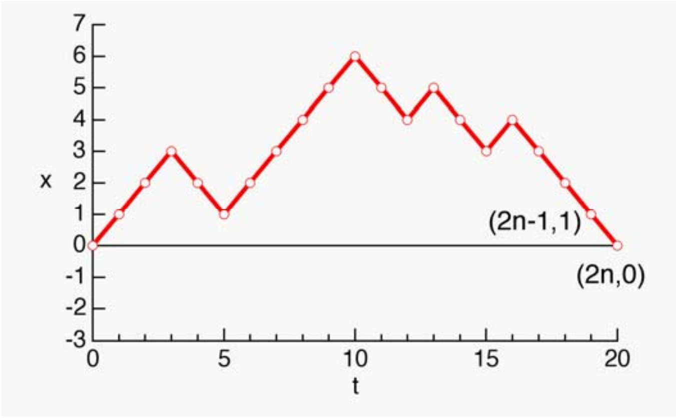
→ Reference book [2], p.94



$$\Rightarrow \frac{x}{t} L(t,x)$$

[Example] How many positive passages from (0,0) to ($2n$,0) ?

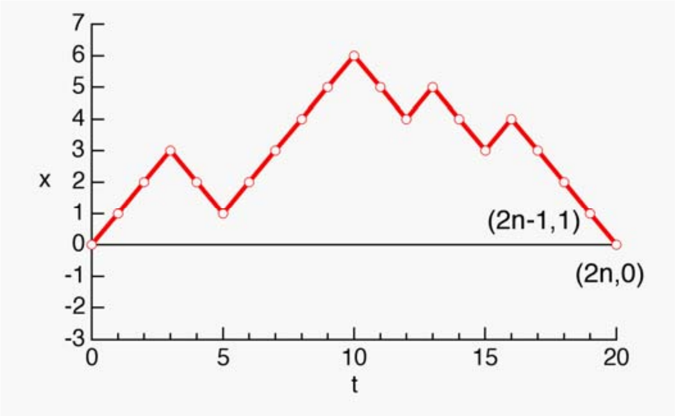
→ Reference book [2], p.97



$$\Rightarrow \frac{1}{2n-1} {}_{2n-1}C_{n-1} = \frac{1}{n} {}_{2n-2}C_{n-1}$$

▪Question of zero return → Reference book [2], p.98

f_{2n} → Probability to return to zero at the time $2n$ for the first time



$$f_{2n} = \frac{1}{2n-1} {}_{2n-1}C_n \left(\frac{1}{2}\right)^{2n-1}$$

$$f_2 = 0.5,\ f_4 = 0.125,\ f_6 = 0.0625$$

v_{2n} → Probability to return to zero not later than the time $2n$

$$v_{2n} = f_2 + f_4 + \cdots f_{2n} = 1 - {}_{2n}C_n 2^{-2n}$$

$$v_6 = 0.6875,\ v_{100} = 0.9204,\ v_{1000} = 0.9748,\ v_{10000} = 0.9920 \Rightarrow v_\infty = 1$$

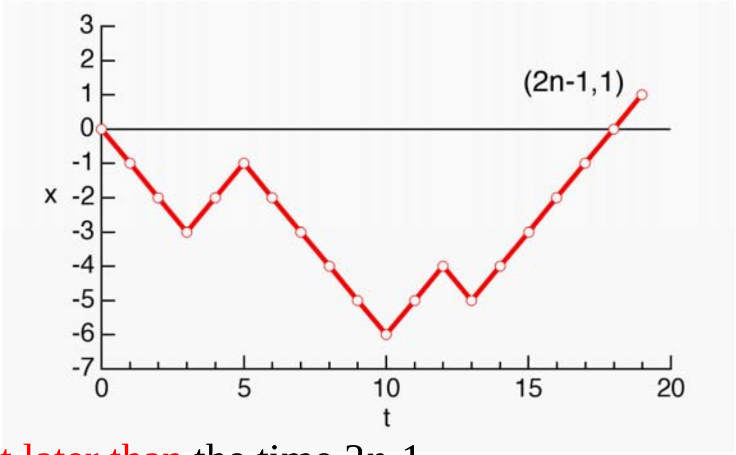
The probability that a particle returns to zero is 1 (ie, the return to zero inevitably happens some time or other).

▪Arrival at level 1

→ Reference book [2], p.102

g_{2n-1} → Probability to arrive at the location $x=1$ at the time $2n$-1 for the first time

$$g_{2n-1} = \underbrace{\frac{1}{2n-1} {}_{2n-1}C_n}_{\text{Drills}} \left(\frac{1}{2}\right)^{2n-1} = f_{2n}$$



w_{2n-1} → Probability to arrive at level 1 not later than the time $2n$-1

$$w_{2n-1} = v_{2n} = 1 - {}_{2n}C_n 2^{-2n} \Rightarrow w_\infty = 1$$

The probability that a particle arrives at level 1 is 1 (ie, the arrival at Level 1 inevitably happens some time or other).

⇨ A particle performing a random walk intersects a given level infinitely, if the probability is 1.（A straight line is to be filled up in the case of one-dimensional random walk .）

A gambler having seed money large enough inevitably gets a gain some time or other; provided that the win/loss probability of the game is 50/50, and that the amount of money of win/loss is the same.

4-3 Random Walk and Diffusion

→ Reference book [3], p.86

- Relation with diffusion phenomenon: Long-time limit, $N >> m$

$$W(N,m) = {}_N C_k \left(\frac{1}{2}\right)^N = {}_N C_{(N-m)/2} \left(\frac{1}{2}\right)^N = \frac{N!}{\left(\frac{N-m}{2}\right)!\left(\frac{N+m}{2}\right)!}\left(\frac{1}{2}\right)^M$$

Binomial distribution

Stirling's formula

$$\ln n! \cong \left(n + \frac{1}{2}\right)\ln n - n + \frac{1}{2}\ln(2\pi)$$

$$\Rightarrow \ln W(N,m) \cong \frac{1}{2}\ln\frac{2}{\pi N} - \frac{N}{2}\left[\left(1 + \frac{m+1}{N}\right)\ln\left(1 + \frac{m}{N}\right) + \left(1 + \frac{1-m}{N}\right)\ln\left(1 - \frac{m}{N}\right)\right]$$

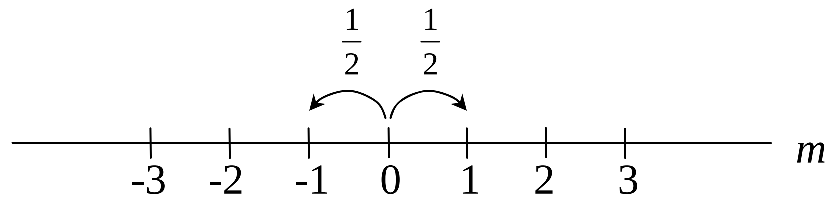
To be expanded in powers of (m/N) up to the second order

$$\ln(1+x) \cong x - \frac{x^2}{2}$$

$$\ln W(N,m) \cong \frac{1}{2}\ln\frac{2}{\pi N} - \frac{N}{2}\left(\frac{m}{N}\right)^2$$

$$\Rightarrow W(N,m) = \sqrt{\frac{2}{\pi N}}\exp\left[-\frac{N}{2}\left(\frac{m}{N}\right)^2\right]$$

Normal distribution



$$W(N,m)=\sqrt{\frac{2}{\pi N}}\exp\left[-\frac{N}{2}\left(\frac{m}{N}\right)^2\right]$$

$$x=ml \qquad t=N\tau$$

$$P(t,x)dx=W\left(\frac{x}{l},\frac{t}{\tau}\right)\frac{dx}{l}\frac{1}{2}=\sqrt{\frac{\tau}{2\pi l^2 t}}\exp\left[-\frac{\tau x^2}{2l^2 t}\right]dx$$

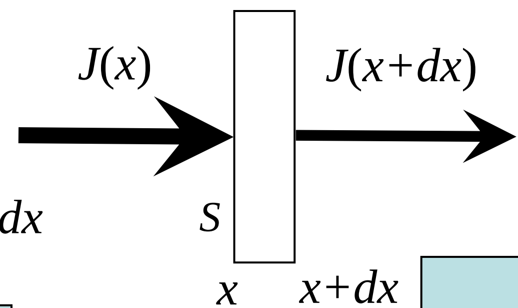
“Concentration”

$$\boxed{\frac{\partial P}{\partial t}=\frac{l^2}{2\tau}\frac{\partial^2 P}{\partial x^2}}$$

- Fick's law pertaining to the diffusion of substances
  - Flux (mass passing through a unit cross-sectional area per unit time) is proportional to concentration gradient

$$J=-D\frac{\partial P}{\partial x}$$

D : Diffusion coefficient



$$[J(x)dt-J(x+dx)dt]S=[P(t+dt)-P(t)]Sdx$$

$$\Rightarrow\quad -\frac{\partial J}{\partial x}=\frac{\partial P}{\partial t}$$

$$\boxed{\frac{\partial P}{\partial t}=D\frac{\partial^2 P}{\partial x^2}}$$

$$\boxed{D=\frac{l^2}{2\tau}}$$

- Random walk and diffusion phenomenon

$$P(t,x)=\sqrt{\frac{1}{4\pi Dt}}\exp\left[-\frac{x^2}{4Dt}\right]$$

Fluctuation-dissipation theorem

Fluctuation (in equilibrium)   Dissipation/ transportation

$$D=\frac{l^2}{2\tau} \Rightarrow \langle x^2\rangle = 2Dt$$

Variance of position σ_x^2

- Initial Conditions
  - What will concentration distribution be at $t = 0$?

$$\left.\begin{aligned}&\int_{-\infty}^{\infty}P(t,x)dx=1\\ &x\neq 0 \Rightarrow P(t\to +0,x)=0\\ &x=0 \Rightarrow P(t\to +0,x=0)=\infty\end{aligned}\right\}$$

Dirac's delta function

$$\delta(x)=\lim_{t\to +0}\sqrt{\frac{1}{4\pi Dt}}\exp\left[-\frac{x^2}{4Dt}\right]$$

⇨ Distribution concentrating on $x = 0$

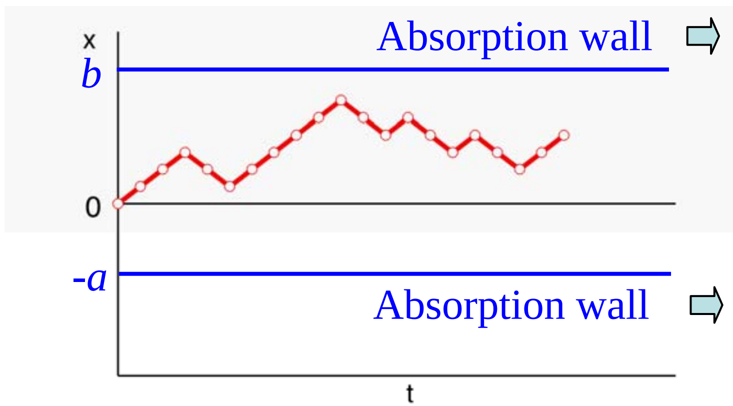
Random walk is a one-dimensional diffusion equation, and

is one of the models of $\dfrac{\partial P}{\partial t}=D\dfrac{\partial^2 P}{\partial x^2}$ $\quad P(t=0,x)=\delta(x)$

4-4 Gambler's Ruin Problem

→ Reference book [2], p.150

- Random walk when absorption wall exists



⇨ Ruin of the gambler having money *b*

⇨ Ruin of the gambler having money *a*

What is the probability that the money runs out at level - *a* before arriving at level *b*? ⇨ p_a

$$p_a = \frac{1}{2} p_{a-1} + \frac{1}{2} p_{a+1} \qquad p_0 = 1 \qquad p_{a+b} = 0$$

Consider on the assumption that a+b (sum total of money) is fixed.

$$\Rightarrow \quad p_a = \frac{b}{a+b}$$

How will it be, if $b \to \infty$?