Chapter 6

Symmetry and Conservation Law

6.1 Symmetrical Operation and Symmetry Operator

Suppose some operations such as a mirror reflection, one motion can be deduced from the equation of motion. If the whole motion was obtained by the operation of original motion, it is called reflected motion. If it also satisfies the same equation of motion, then we call it "symmetrical operation".

If $\widehat{T}$ fell into such operation and $|\psi(t)\rangle$ was a solution to the equation of motion, while the other motion $\widehat{T}|\psi(t)\rangle$ satisfies the operation, it is considered to be symmetrial. Under the condition, $i\hbar\, d\widehat{T}|\psi(t)\rangle/dt = \widehat{H}\widehat{T}|\psi(t)\rangle$, the left hand side can be rewritten as $i\hbar\, d\widehat{T}|\psi(t)\rangle/dt = \widehat{T}\, i\hbar\, d|\psi(t)\rangle/dt = \widehat{T}\widehat{H}|\psi(t)\rangle$ based on the fact that $|\psi(t)\rangle$ satifsies the original equation of motion.

When the operation is symmetrical, we can conclude that,

$$\widehat{H}\widehat{T} = \widehat{T}\widehat{H}.$$

This relation is often said in a way that two operators $\widehat{H}$ and $\widehat{T}$ "commutes" each other.

6.2 Conservation Law

It is proved by the matrix mathematics, that all the eigen states of two commuting operators stay in common. In degeneration, we can deliverately select the common eigen states. At any eigen states of the Hamiltonian operator, $\widehat{H}$ is stationary, thus any eigen states of the symmetrical operator can be stationary.

- Eigen value of any symmetrical operator is conserved. (Conservation Law)

6.3 Bi-Symmetry and Parity

There are many bi-symmetrical operations; repeating the same operations twice results in identical to the initial state, which include mirror reflection, origin symmetry, particle exchange, etc. Apparently, $\widehat{T}^2 = \widehat{I}$. Suppose the eigen value of $\widehat{T}$ is t, then the relations $t^2 = 1$ is derived and we get the eigen value $t = \pm 1$. If $\widehat{T}$ was the symmetrical operator, then these eigen values should be conserved. We call $t = 1$ an even parity, and $t = -1$ an odd parity.

Exchanging the operations of two identical integer-spin particles keeps the parity in even, and derives Bose-Einstein statics. When the operation is exchanged by the two identical half-integer-spin particles that keep odd parity, then it derives Fermi-Dirac statistics.

6.4 Transversal Symmetry and Momentum

If there was no notch or potential variations in a tranversal motion of a particle, the transversal operator $\widehat{T}(x)$ corresponding to the leftside motion of x becomes symmetrical and any eigen value of $\widehat{T}(x)$ will be conserved. Like the deduction of Hamiltonian $\widehat{H}(t)$ from the time progress operator $\widehat{U}(t, t')$, we can define an infinitesimal operator $-i\hbar\, d\widehat{T}(x)/dx$, with its eigen value, which is also conserved. This operator is apparently the momentum operator $\widehat{p}$, therefore, again, the momentum can be conserved.

6.5 Rotational Symmetry and Angular Momentum

Based on the observation of the rotational operator $\widehat{R}_z(\theta)$ corresponding to the left turn θ along z axis, an angular momentum operator J_z can be deduced. Although, rotation operations are not additional to the bigger rotation, the rotational operator along x axis and that along y axis do not commute each other.

$$[\widehat{R}_x(\theta), \widehat{R}_y(\phi)] = \widehat{I} - \widehat{R}_z(\theta\phi) + O^3.$$

This piece of information tells us so much:

$$[\widehat{J}_x, \widehat{J}_y] = i\hbar\widehat{J}_z$$
$$[\widehat{J}_y, \widehat{J}_z] = i\hbar\widehat{J}_x$$
$$[\widehat{J}_z, \widehat{J}_x] = i\hbar\widehat{J}_y,$$

these are the most important characterisitics of the angular momentum.

- Square of a total angular momentum takes the value $j(j+1)\hbar^2$, when j is integer or half integer positive number.

- If we measured an angular momentum along certain fixed axis of the total angular momentum the values are detected between the minima $-j\hbar$ and the maxma $j\hbar$ at an interval of $\hbar$.

6.6 Hydrogen Atom

Abbreviated.