Chapter 3

States and Operator

3.1 Base States and its Ortho-Normality and Completeness

A set of the states that satfisfies the following relation is called “base states”.

$$\langle j | k \rangle = \delta_{jk} \qquad \text{(Ortho-Normal)}$$

$$|\psi\rangle = \sum_{\text{all } j} |j\rangle \langle j | \psi \rangle \qquad \text{(Complete).}$$

$|x\rangle , |y\rangle$ are the base states from the polarized light. And there are some other sets of base states, also. For example,

$$|45^\circ\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle), \qquad |-45^\circ\rangle = \frac{1}{\sqrt{2}}(|x\rangle - |y\rangle).$$

3.2 Polarized States

Any polarizations can be separated into two states $|x\rangle$ and $|y\rangle$ by Nicol prism.

Therefore, we can say that any polarized state can be defined by a combination of a set of base states $\{|x\rangle , |y\rangle\}$. Such as,

$$|\theta\rangle = \cos\theta\, |x\rangle + \sin\theta\, |y\rangle)$$

$$|45^\circ\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)$$

$$|-45^\circ\rangle = \frac{1}{\sqrt{2}}(|x\rangle - |y\rangle)$$

$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle) \qquad \text{(Right Hand Circular)}$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle) \qquad \text{(Left Hand Circular).}$$

Furthermore, $\{|45^\circ\rangle , |-45^\circ\rangle\}$ and $\{|R\rangle , |L\rangle\}$ are the sets of base states respectively.

3.3 Spin 1/2 States

Electrons are considered to be 1/2 particles, because the magnitude of the spin is 1/2 times $\hbar$, and the particles are separated into two states $|+z\rangle$ and $|-z\rangle$ in an inhomogeneous magnetic field B_z, in which $\{|+z\rangle, |-z\rangle\}$ is a set of base states, and any spin 1/2 electron states can be represented by a combination of these states.

$$|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$$

$$|-x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle)$$

$$|+y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + i\,|-z\rangle)$$

$$|-y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - i\,|-z\rangle).$$

Furthermore, $\{|+x\rangle, |-x\rangle\}$ and $\{|+y\rangle, |-y\rangle\}$ are consisted of other base states respectively.

3.4 Particles in One-Dimensional Space

Any single particle without consideration of any inner state such as spin can be expressed in the combination of position base states $\{|x_j, y_k, z_l\rangle\}$ by cutting the whole space periodically with a pitch of $\Delta x, \Delta y, \Delta z$.

In one-dimensional space, each base state in above becomes $\{|x_j\rangle\}$, which corresponds to the space $x_j \sim x_j + \Delta x$.

The discussion becomes simpler when the total size of the space is limited by $-L/2 \sim L/2$ and $L = N\Delta x$, as well as the periodical boundary condition is assumed.

Apparently

$$\langle x_j | x_k \rangle = \delta_{jk},$$

and

$$|\psi\rangle = \sum_{\text{all } j} |x_j\rangle \langle x_j | \psi \rangle .$$

3.5 Momentum Base States

As one of the one-dimensional states under the periodic boundary condition, there exists a momentum state in the following:

$$\langle x_j | p_J \rangle = \frac{1}{\sqrt{N}} \exp\left(i \frac{p_J x_j}{\hbar} \right).$$

p_J is $p_J = (2\pi\hbar/L)J$ where J is an integer in the region $-(N-1)/2 \leq J \leq (N-1)/2$. This set of momentum states can construct another set of the base states.

3.6 Continuous One-Dimensional Space

By setting $\Delta x \to 0$, the one-dimensional space can be made continuous. In this way, probability density $p(x_j) = P(x_j \sim x_j + \Delta x)/\Delta x$ is prefered over the probability $P(x_j \sim x_j + \Delta x)$.

According to the above treatment, the probability amplitude is replaced by the probability density amplitude.

$$\langle \underline{x} | \, \underline{x'} \rangle = \delta_{xx'}/\Delta x \qquad \text{(Ortho-Normal)}$$

By $\Delta x \to 0$, the right-most function will be Dirac δ function $\delta(x - x')$.

$$\psi = \sum \Delta x \, |\underline{x}\rangle \langle \underline{x} | \, \psi \rangle \qquad \text{(Complete).}$$

By $\Delta x \to 0$, $\sum \Delta x$ becomes $\int dx$. $\langle \underline{x} | \, \psi \rangle$ is sometimes called wave function $\psi(x)$.

3.7 Operator

Any process or apparatus that is capable of changing the quantum state can be expressed by a operator. Apparently we suppose linear processes.

$$\widehat{A} \, |\psi\rangle = |\phi\rangle \, .$$

We can expand the operator by basic states. $\widehat{\ }$ is attached for the clarification that A is not a scalar variable.

$$\langle j | \, \widehat{A} \, |k\rangle \langle k | \, \psi \rangle = \langle k | \, \phi \rangle \, .$$

[Example] x polarizer

$$(x \text{ polarizer}) \, |x\rangle = |x\rangle \, , \qquad (x \text{ polarizer}) \, |y\rangle = 0$$

From the above facts, we can deduce

$$(x \text{ polarizer}) \, |\psi\rangle = (x \text{ polarizer}) \, |x\rangle \langle x | \, \psi \rangle + (x \text{ polarizer}) \, |y\rangle \langle y | \, \psi \rangle = |x\rangle \langle x | \, \psi \rangle \, .$$

3.8 Identity Operator

An operation which does not involve any changes in state, we define identity operator $\widehat{I}$.

$$\widehat{I} \, |\psi\rangle = |\psi\rangle$$

$$\langle j | \, \widehat{I} \, |k\rangle = \langle j | \, k \rangle = \delta_{jk}.$$

Based on the facts, the completeness equation can be expressed as follows:

$$\sum_{\text{all } j} |j\rangle \langle j| = \widehat{I} \qquad \text{(Complete).}$$

By including the equations between the bra and ket, an expansion by the basic states can be easily obtained.