June 11 Lecture Schedule

Differential and integral calculus of the power series $\sum_{n=0}^{\infty} a_n x^n$.

Theorem. Let r be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ in which we can hold the following statements:

1. The power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also have the radius of convergence r.

2. If we define the function $f(x)$ as $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $-r < x < r$, then $f(x)$ is differentiable on $-r < x < r$ so that we can write as

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \int_0^x f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

.

[Proof] 1. Let r' be the radius of convergence of $\sum_{n=1}^{\infty} n a_n x^{n-1}$.

(1) $n a_n x^{n-1} \to 0 (n \to \infty)$ when $|x| < r$

(2) $a_n x^n \to 0 (n \to \infty)$ when $|x| < r'$

Showing these two may suffice for the proof. We begin by showing (2). If $|x| < r'$ we can write as $a_n x^n \to 0 (n \to \infty)$ thereby $a_n x^{n-1} \to 0 (n \to \infty)$. Now, we show (1).

For $|x| < t < r$, N that satisfies $|a_n t^n| < 1$ if $n \geq N$ must exist such that

$$|n a_n x^{n-1}| \leq n(\frac{|x|}{t})^{n-1} t \to 0 \ (n \to \infty)$$

2. We let $f_n(x) = \sum_{k=0}^{n} a_k x^k$.

Continuation. We let $|x|, |a| \leq s < t < r$ and take N for which to hold $|a_n| t^n \leq 1$ when $n \geq N$. If $n \geq N$, we can write

$$
\begin{aligned}
|f(x)-f_n(x)| &\le \sum_{k=n+1}^{\infty} |a_k x^k| \le \sum_{k=n+1}^{\infty} |a_k| s^k = \sum_{k=n+1}^{\infty} |a_k| t^k \left(\frac{s}{t}\right)^k \\
&\le \sum_{k=n+1}^{\infty} \left(\frac{s}{t}\right)^k = \left(\frac{s}{t}\right)^{n+1} \frac{1}{1-\frac{s}{t}} (= \varepsilon_n \text{ in this case})
\end{aligned}
$$

In the same manner, we can express for $x = a$. Therefore,

$$
\begin{aligned}
|f(x)-f(a)| &\le |f(x)-f_n(x)| + |f_n(x)-f_n(a)| + |f(a)-f_n(a)| \\
&\le |f_n(x)-f_n(a)| + 2\varepsilon_n
\end{aligned}
$$

Thus,

$$
\lim_{x\to a} |f(x)-f(a)| \le 2\varepsilon_n \to 0 \quad (n\to\infty).
$$

Integration. Take $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ and $|x| \le t < r$, which provide

$$
\int_0^x f(x)dx = \int_0^x f_n(x)dx + \int_0^x (f(x)-f_n(x))dx
$$

Thus,

$$
|\int_0^x (f(x)-f_n(x))dx| \le \int_0^x |f(x)-f_n(x)|dx \le \varepsilon_n |x| \to 0 \quad (n\to\infty).
$$

Differentiation. We can apply the result provided above to the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$

The error term estimation.

In Taylor 's theorem:

$$
f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1}
$$

In power series expansion:

$$
\sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{n} a_k x^k = \sum_{k=n+1}^{\infty} a_k x^k
$$

The estimation for majorant series.

Alternating series convergence ([1] p.188 Theorem 8, [2] p.24 Theorem 1.4, [3] p.45 Theorem 1.23)

$\sum_{n=0}^{\infty}(-1)^n a_n$ is convergent if it is monotonically decreasing and $a_n > 0, \lim_{n\to 0} a_n = 0$.

$$|a - \sum_{k=0}^{n}(-1)^k a_k| \leq |a_{n+1}|.$$

The power series expansion for other functions. ([1] p.44 Example 20(5), p.194 Exercise 3, [3] p.239 Example 5.11)

$$\begin{aligned}(1+x)^a &= \sum_{n=0}^{\infty}\binom{a}{n}x^n \\ &= 1 + ax + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \frac{a(a-1)(a-2)(a-3)}{4!}x^4 + \cdots .\end{aligned}$$