June 4 Lecture Schedule

The radius of convergence calculation ([1] p.192, [2] II p.133,)

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = l$ (or $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = l$) exists then $r = \frac{1}{l}, (l \neq 0)$, and $r = \infty, (l = 0)$ can be given.

[Proof] If let $|x| < 1/l$ we can write $\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| < 1.$ from which $\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| > 1$ must be provided if $a_n x^n \to 0 (n \to \infty)$, and $x > 1/l$ therefore $|a_n x^n| \to \infty (n \to \infty)$ is obtained.

Example

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

Differential and integral calculus of the power series $\sum_{n=0}^{\infty} a_n x^n$ ([1] p.193, [2] II p.134-136, [3] p.233-242.)

Theorem Let r be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. In such a case, the following statements can be held:

1. The power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$ have radii of convergence r.

2. If we define the function $f(x)$ as $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $-r < x < r$, $f(x)$ is differentiable on $-r < x < r$ so that we can write as

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \int_0^x f(x)dx = \sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$$

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Example.

$$\begin{aligned}\log(1+x) &= \int_0^x \frac{1}{1+x}dx = \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots, \\ \text{Arctan}x &= \int_0^x \frac{1}{1+x^2}dx = \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right)dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots.\end{aligned}$$

[Proof] 1. Let r' be the radius of convergence of $\sum_{n=1}^{\infty} na_n x^{n-1}$.

(1) $na_n x^{n-1} \to 0 (n \to \infty)$ stands when $|x| < r$

(2) $a_n x^n \to 0 (n \to \infty)$ stands when $|x| < r'$

Showing these two may suffice for the proof. Let us begin by showing (2). We have $a_n x^n \to 0 (n \to \infty)$ if $|x| < r'$ so that $a_n x^{n-1} \to 0 (n \to \infty)$ can be given. Now, we show (1). There is existing N which satisfies $|a_n t^n| < 1$ if $n \geq N$ at $|x| < t < r$, which gives

$$|na_n x^{n-1}| \leq n(\frac{|x|}{t})^{n-1} t \to 0 \ (n \to \infty)$$

2. We let $f_n(x) = \sum_{k=0}^{n} a_k x^k$.

Continyuation. Let $|x|, |a| \leq s < t < r$. We take N for which to satisfy $|a_n|t^n \leq 1$ at $n \geq N$. So, if $n \geq N$ we can write

$$\begin{aligned}|f(x) - f_n(x)| &\leq \sum_{k=n+1}^{\infty} |a_k x^k| \leq \sum_{k=n+1}^{\infty} |a_k| s^k = \sum_{k=n+1}^{\infty} |a_k| t^k \left(\frac{s}{t}\right)^k \\ &\leq \sum_{k=n+1}^{\infty} \left(\frac{s}{t}\right)^k = \left(\frac{s}{t}\right)^{n+1} \frac{1}{1-\frac{s}{t}} (= \varepsilon_n \text{とおく})\end{aligned}$$

Likewise, we may express the case $x = a$. Thus,

$$\begin{aligned}|f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f(a) - f_n(a)| \\ &\leq |f_n(x) - f_n(a)| + 2\varepsilon_n\end{aligned}$$

such that,

$$\lim_{x\to a}|f(x)-f(a)| \leq 2\varepsilon_n \to 0 \quad (n\to\infty).$$

Here, note that the inequality

$$|f(x)-f_n(x)| \leq \varepsilon_n$$

can be given independent of$x$. In such a case, the functional sequence $f_n(x)$ is known to be uniformly convergent to $f(x)$.

[Another proof of the power series differential and integral calculus theorem 2]

$$\begin{aligned}
|f(x)-f(a)| &= |\sum_n a_n(x^n-a^n)| \leq \sum_n |a_n||x^n-a^n| \\
&= |x-a|\sum_n |a_n||x^{n-1}+x^{n-2}a+\cdots+a^{n-1}| \\
&\leq |x-a|\sum_n n|a_n|t^{n-1} \to 0 \quad (x\to a)
\end{aligned}$$

This is differentiable. We let $g(x)=\sum_{n=1}^{\infty} na_n x^{n-1}$ and $|x|,|a|\leq t<r$:

$$\begin{aligned}
&\frac{f(x)-f(a)}{x-a}-g(a)=\sum_n a_n(\frac{x^n-a^n}{x-a}-na^{n-1}) \\
&= \sum_n a_n((x^{n-1}-a^{n-1})+(x^{n-2}a-a^{n-1})+\cdots+(xa^{n-2}-a^{n-1})) \\
&= (x-a)\sum_n a_n((x^{n-2}+\cdots+a^{n-2})+(x^{n-3}a+\cdots+a^{n-2})+\cdots+a^{n-2})
\end{aligned}$$

Thus,

$$\left|\frac{f(x)-f(a)}{x-a}-g(a)\right| \leq |x-a|\sum_n \frac{n(n-1)}{2}|a_n|t^{n-2} \to 0 \quad (x\to a)$$