May 21 Lecture Schedule

Power series expansion of e^x ([1] p.42 Example 18, p.44 Example 20, [2] p.85, [3] p.145 Example 3.9)

$$e^x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) = \frac{e^t \cdot x^{n+1}}{(n+1)!}.$$

The absolute value on the right side of the equation above is given by $\leq \frac{e^{|x|} \cdot |x|^{n+1}}{(n+1)!}$, in which we have $\to 0$ at $n \to \infty$. Thus,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

, which we call the power series expansion of e^x. In the same manner, we can determine the power series expansion

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

.

Using the summation formula of the geometric series, we can write down as in the following for $|x| < 1$:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

.

We integrate the equation above termwise, and try to obtain

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

In general, the series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ for the sequence $a_0, a_1, a_2, \ldots$ is called the power series.

The convergence of power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ ([1] pp.190-192, [2] II pp. 131-133, [3] pp.231-232)

Theorem. Take a set of real numbers $S \subset T$ and define

$$S = \{x \in \mathbf{R} | \sum_{n=0}^{\infty} a_n x^n \text{is convergent}\}$$

,and

$$T = \{x \in \mathbf{R} | a_n x^n \to 0 \ (atn \to \infty)\}$$

1. Whichever of the facts below must be held for T

(1) The real number r≥ 0 exists while which satisfies $T = [r, -r]$.

(1') The real number r≥ 0 exists while which satisfies $T=(r,-r)$ (2) $T=\mathbf{R}$.

2. One of the facts below must be held for S.

(1) $(r,-r)\subset S\subset[r,r]$ must be given if $T=[r,-r]$ or $(r,-r)$

(2) $S=\mathbf{R}$ when $T=\mathbf{R}$.

Definition. Under(1), r is called the radius of convergence for the power series $\sum_{n=0}^{\infty}a_nx^n$. For (2), we call the radius of convergence for $\sum_{n=0}^{\infty}a_nx^n$ is ∞.

Example. The radius of convergence for the power series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ is ∞. The radius of convergence for $\sum_{n=0}^{\infty}x^n$ is 1.

[Proof] 1. Note that $T=-T$. If and only $0\leq x<t$ and $t\in T$ then we can write $x\in T$. So, we can let $r=\sup T$ if T is bounded thereby, either (1) or (1') can be given while (2) can be given if T is not bounded.

2. It is clear that $S\subset T$. We now let $t\in T$.

By making the natural number N large enough, we find that $|a_nt^n|\leq 1$ stands for all $n\geq N$. If we let $|x|<t$ we find that $|a_nx^n|\leq\left(\frac{|x|}{t}\right)^n$ stands for $n\geq N$ and thus, the majorant series method is used to learn that $\sum_{n=0}^{\infty}a_nx^n$ is absolutely convergent. So, $(-t,t)\subset S$ is given.

Thus, $\bigcup_{n=1}^{\infty}(-r+\frac{1}{n},r-\frac{1}{n})=(-r,r)\subset S\subset[-r,r]$is provided when (1) while $\bigcup_{n=1}^{\infty}(-n,n)=\mathbf{R}\subset S\subset\mathbf{R}$ is given when (2).

The calculation for the radius of convergence ([1] p.192, [2] II p.133)

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=l$ (or $\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=l$) exists, then $r=\frac{1}{l},(l\neq 0)$, and $r=\infty,(l=0)$ can be obtained.

[Proof] If we let $|x|<1/l$ we can write $\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right|<1$. so that if we let $a_nx^n\to 0(n\to\infty)$, $x>1/l$ then we have $\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right|>1$) thus, $|a_nx^n|\to\infty(n\to\infty)$.