**May 14** Lecture Schedule
Taylor 's theorem ([1] p.40 Theorem 11, [2] p.83-84, [3] p.132 Theorem 3.14)
$t$ that exists between $a$ and $x$ satisfies

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) &+ f''(a)\frac{(x-a)^2}{2} + f^{(3)}(a)\frac{(x-a)^3}{3!} + \cdots \\ &+ f^{(n)}(a)\frac{(x-a)^n}{n!} + f^{(n+1)}(t)\frac{(x-a)^{n+1}}{(n+1)!} \end{aligned}$$

Proof. Consider a case for $x \geq a$. We take

$$F(x) = f(x) - \left( f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} \right)$$

.

Knowing that $F(a) = F'(a) = \cdots = F^{(n)}(a) = 0, F^{(n+1)}(x) = f^{(n+1)}(x)$, we repeat generalization of the mean-value theorem and repeat applying it to find $x \geq x_1 \geq x_2 \cdots x_{n+1} = t \geq a$ that satisfies

$$\begin{aligned} \frac{F(x)}{(x-a)^{n+1}} &= \frac{F'(x_1)}{(n+1)(x_1-a)^n} = \frac{F^{(2)}(x_2)}{(n+1)n(x_2-a)^{n-2}} = \cdots \\ &= \frac{F^{(n)}(x_n)}{(n+1)!(x_n-a)} = \frac{F^{(n+1)}(x_{n+1})}{(n+1)!} = \frac{f^{(n+1)}(t)}{(n+1)!} \end{aligned}$$

Before we go to the final step, we take the limit of $x \to a$, and which gives us

$$\lim_{x \to a} \frac{F(x)}{(x-a)^{n+1}} = \frac{F^{(n+1)}(a)}{(n+1)!} = \frac{f^{(n+1)}(a)}{(n+1)!}$$

Rewrite the above by replacing $n+1$ with $n$, we obtain

$$\lim_{x \to a} \frac{f(x) - \left( f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} \right)}{(x-a)^n} = 0$$

For example,

$$\lim_{x \to 0} \frac{\sin x - \left( x - \frac{x^3}{6} \right)}{x^4} = 0$$

This can be also expressed as

$$\sin x - \left( x - \frac{x^3}{6} \right) = o(x^4)$$

.

Interpretations of the Taylor 's rule.
$f(x)$ : true value

$f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2}+\cdots+f^{(n)}(a)\frac{(x-a)^n}{n!}$ : approximate value
$f^{(n+1)}(t)\frac{(x-a)^{n+1}}{(n+1)!}$ : error
For example,

$$e-\left(1+1+\frac{1}{2}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)=\frac{e^t}{(n+1)!}\leq\frac{e}{(n+1)!}.$$

Recall that

$$\begin{aligned} e-\left(1+1+\frac{1}{2}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right) &= \sum_{k=n+1}^{\infty}\frac{1}{k!} \\ < \quad \frac{1}{(n+1)!}\frac{1}{1-1/(n+2)} &= \frac{1}{(n+1)!}\left(1+\frac{1}{n+1}\right). \end{aligned}$$

The integral form of remainder terms ([2] p.85)

$$\begin{aligned} f(x) &= f(a)+\int_a^x f'(t)dt \\ &= f(a)-[(x-t)f'(t)]_a^x+\int_a^x (x-t)f''(t)dt \\ &= f(a)+(x-a)f'(a)-[\frac{(x-t)^2}{2}f^{(2)}(t)]_a^x+\int_a^x \frac{(x-t)^2}{2}f^{(3)}(t)dt \\ &\cdots \\ &= f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2}+\cdots+f^{(n)}(a)\frac{(x-a)^n}{n!} \\ &\quad +\int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)dt \end{aligned}$$