May 7 Lecture Schedule

What is the differential calculus? First order approximation.

Definition.

$$\frac{f(x+h)-(f(x)+f'(x)h)}{h} \to 0 \qquad (h \to 0).$$

Mean value theorem ([1] p.38 Theorem 9, [2] p.77 Theorem 3.2, [3] p.119 Theorem 3.5)

α that satisfies

$$f(x+h) = f(x) + f'(\alpha)h$$

must exist.

Generalization. If $g'(x) \neq 0$ then c must exist between a and b, and which satisfies

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

.

Proof Let $F(x) = (g(b)-g(a))(f(x)-f(a)) - (f(b)-f(a))(g(x)-g(a))$, giving $F(a) = F(b) = 0$. So that c must exist between a and b and which satisfies $F'(c) = 0$. Note that $F'(x) = (g(b)-g(a))f'(x) - (f(b)-f(a))g'(x)$.

$a \leq c \leq b$ that satisfy the integral form of the mean value theorem $\int_a^b f(x)dx = f(c)(b-a)$ must exist.

Let the maximum value be M and the minimum value be m to write $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$. Using the intermediate value theorem, we must have $a \leq c \leq b$ which satisfies $f(c) = \int_a^b f(x)dx/(b-a)$.

Taylor's theorem

High-order derivatives ([1] pp.32-35, [2] pp.95-97, [3] pp.127-132)

e^x, $\sin x$, etc.

Leipniz' rule ([1] p.34 Theorem 6, [2] p.96 Theorem 3.6, [3] p.127 Theorem 3.11)

$$(fg)^{(n)} = \sum_{p+q=n} \binom{n}{p} f^{(p)} g^{(q)}.$$

Taylor 's theorem ([1] p.40 Theorem 11, [2] pp.83-84, [3] p.132 Theorem 3.14) t, existing between a and x, satisfies

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) &+ f''(a)\frac{(x-a)^2}{2} + f^{(3)}(a)\frac{(x-a)^3}{3!} + \cdots \\ &+ f^{(n)}(a)\frac{(x-a)^n}{n!} + f^{(n+1)}(t)\frac{(x-a)^{n+1}}{(n+1)!} \end{aligned}$$

Proof. Consider a case for $x \geq a$. We let

$$F(x) = f(x) - \left(f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} \right)$$

Given $F(a) = F'(a) = \cdots = F^{(n)}(a) = 0$, we repeat generalization of the mean-value theorem and repeat applying the generalized theorem to obtain $x \geq x_1 \geq x_2 \cdots x_{n+1} = t \geq a$, which satisfies

$$\begin{aligned}\frac{F(x)}{(x-a)^{n+1}} &= \frac{F'(x_1)}{(n+1)(x_1-a)^n} = \frac{F^{(2)}(x_2)}{(n+1)n(x_2-a)^{n-2}} = \cdots \\ &= \frac{F^{(n+1)}(x_{n+1})}{(n+1)!} = \frac{f^{(n+1)}(t)}{(n+1)!}\end{aligned}$$

Integral form of the remainder terms ([2] p.85) ([2] p.85)

$$\begin{aligned}f(x) &= f(a) + \int_a^x f'(t)dt \\ &= f(a) - [(x-t)f'(t)]_a^x + \int_a^x (x-t)f''(t)dt \\ &= f(a) + (x-a)f'(a) - [\frac{(x-t)^2}{2}f^{(2)}(t)]_a^x + \int_a^x \frac{(x-t)^2}{2}f^{(3)}(t)dt \\ &\cdots \\ &= f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} \\ &\quad + \int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)dt\end{aligned}$$