April 30 Lecture Schedule

Intermediate value theorem([1] p.11 Theorem7, [2] p.44 Theorem2.5, [3] p.82 Theorem2.2)

Let $f(x)$ be a continuous function defined on $a \leq x \leq b$. If c satisfies $f(a) \leq c \leq f(b)$ then a real number exists while it satisfies $f(x) = c$,and $a \leq x \leq b$.

Maximum value theorem([1] p.36 Theorem 7, [2] p.45 Theorem 2.6, [3] p.85 Theorem 2.4)

The continuous function $f(x)$ defined on $a \leq x \leq b$ holds maximum values at $a \leq x \leq b$. Thus, a real number must exist and which satisfies $a \leq c \leq b$ and $f(x) \leq f(c)$ for an arbitrary real number $a \leq x \leq b$.

The restatement of the continuity of real numbers, No. 3 ([1] p.2, [2] p.48 Theorem 2.7, [3] p.36)

The supermum exists if a set of real numbers S is bounded and non-empty.

Bounded above: a natural number M which satisfies $x \leq M$ for the arbitrary $x \in S$ must exist.

The supermum: the meaning in which $s = \sup S$ being the supermum of S is that $x \leq s$ stands for the arbitrary $x \in S$ and s denotes the limit of the elements of S. This indicates that there is a sequence $x_n \in S; n = 1, 2, 3, \ldots$ consisted of the elements of S, and which satisfies $s = \lim_{n\to\infty} x_n$.

The above restatement No.3$\Rightarrow$ Intermediate value theorem Let $S = \{a \leq x \leq b | f(x) \leq c\}$. S is non-empty because $a \in S$. If $x \in S$, then S is bounded because of $x \leq b$.

Now let $s = \sup S$ to show $f(s) = c$. If $x > s$ we can write $f(x) > c$ because of $x \notin S$ thereby $f(s) \geq c$. Here, if we let $\lim_{n\to\infty} x_n = s, x_n \in S$ then we may write $f(s) = \lim_{n\to\infty} f(x_n) \leq c$ since $f(x_n) \leq c$. .Hence, giving $f(s) = c$.

The restatement of the continuity of real numbers, No.4

A common part $\bigcap_{n=1}^{\infty} S_n$ of a decreasing sequence $S_n, n = 1, 2, \ldots$ which is non-empty and bounded closed set, must be non-empty.

Boundedness: there is a natural number M which satisfies $-M \leq x \leq M$ for the arbitrary $x \in S$.

Closed set: if a sequence x_n consisted of the elements of S is convergent, the limit of the sequence must always exist inside S. The set is closed through the operation of identifying the limit.

Proposition: for S to be a set that is being closed, it is necessary and sufficient that a continuous function satisfying $S = \{x | f(x) \leq 0\}$ to exist.

[Proof] Necessary. Let $S = \{x | f(x) \leq 0\}$. If $x = \lim_{n\to\infty} x_n, x_n \in S$ then $f(x) = \lim_{n\to\infty} f(x_n) \leq 0$. Hence, $x \in S$.

Sufficient Let $f(x) = \inf\{|x - y| | y \in S\}$.

The restatement, No. 4$\Rightarrow$ Maximum value theorem We let $T = \{f(x) | a \leq x \leq b\}$ and show that T is bounded above. Let us write down $S_n = \{a \leq x \leq b | f(x) \geq n\}$ where S_n is the bounded closed set and therefore, $\bigcap_{n=1}^{\infty} S_n = \emptyset$ can be given. So, n that satisfies $S_n = \emptyset$ must exist. Thus, T must be bounded. Now let $t = \sup T$ and show that $a \leq x \leq b$ exists and satisfies $t = f(x)$. To do so, we let $S_n = \{a \leq x \leq b | f(x) \geq t - \frac{1}{n}\}$. Given $S_n \neq \emptyset$, we can write $\bigcap_{n=1}^{\infty} S_n = \emptyset$. We can let $x \in \bigcap_{n=1}^{\infty} S_n$.

Maximum value theorem ⇒ The restatement, No. 4

Let $a = 0$,and $b = 1$. We take the continuous function $f_n(x)$ which satisfies $0 \leq f_n(x) \leq 1/2^n$ and $S_n = \{0 \leq x \leq 1 | f_n(x) = 1/2^n\}$ for $0 \leq x \leq 1$. (Note that we can use $f_n(x) = (1 - \inf\{|x - y||y \in S\})/2^n$.) If we let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ then $f(x)$ is the continuous function that satisfies $f(x) \geq 1$ so that $S_n = \{x|f(x) \geq 1 - 1/2^n\}$ can be given. The maximum value for $f(x)$ thus becomes 1; however, if $f(x) = 1$ the value becomes $x \in \bigcap_{n=1}^{\infty} S_n$.

The restatement of the continuity of real numbers, No. 2 (Review) The monotonically increasing sequence which is bounded above is convergent.

The restatement, No. 2⇒ The restatement, No. 3

Let M be the upper bound of S. We let $M = M_0$ by taking the elements x_0 of S. We can inductively define x_{n+1} and $M_{n+1} = M_n$ if elements x of S that satisfy $x \geq (x_n + M_n)/2$ exist. While if no such elements exist, we can take (92) and (93). In those cases x_n denotes the monotonically increasing sequence consisting the elements of S, and $s = \lim_{n\to\infty} x_n = \lim_{n\to\infty} M_n$ becomes the supermum of S.

The restatement, No. 3⇒ The restatement, No. 2 If we let $S = \{a_n|n = 1, 2, \ldots\}$ then $\sup S = \lim_{n\to\infty} a_n$.

The restatement, No. 3⇒ The restatement, No. 4 If we let $a_n = \min S_n$ and $s = \lim_{n\to\infty} a_n$, then $s \in \bigcap_{n=1}^{\infty} S_n$.

The restatement, No. 4⇒ The restatement, No. 2 Take $S_n = \{a_m|m \geq n\}$. S_n may become a closed set in the absence of $\lim_{n\to\infty} a_n$. Note that $\bigcup_{n=1}^{\infty} S_n$ is non-empty. Given $s \in \bigcup_{n=1}^{\infty} S_n$ we can write as $s = \lim_{n\to\infty} a_n$.