January 21 Lecture Schedule

Integration of rational functions ([Takahashi, Kato] p.78, 3.3, [Kaneko] I p.113, [Kodaira] n/a)

We can conduct differentiation to confirm $\int \frac{1}{\sqrt{x^2-1}}dx = \log(x+\sqrt{x^2-1})$.:

$$\frac{d}{dx}\log(x+\sqrt{x^2-1}) = \frac{x+\sqrt{x^2-1}}{1+\frac{2x}{2\sqrt{x^2-1}}} = \frac{1}{\sqrt{x^2-1}}.$$

To be able to write the equation, we let $\int f(x,\sqrt{x^2-1})dx$, $x = \frac{t^2+1}{2t} = \frac{1}{2}(t+\frac{1}{t}), 1 < t$ and change the variables. $y = \frac{t^2-1}{2t}$, $\frac{dx}{dt} = \frac{1}{2}(1-\frac{1}{t^2}) = \frac{t^2-1}{2t^2}$, $t = x+y = x+\sqrt{x^2-1}$ thus,

$$\int f(x,\sqrt{x^2-1})dx = \int f(\frac{1+t^2}{2t}, \frac{1-t^2}{2t})\frac{t^2-1}{2t^2}dt.$$

In case where $f(x,y) = \frac{1}{y}$,

$$\int \frac{1}{\sqrt{x^2-1}}dx = \frac{2t}{t^2-1}\frac{t^2-1}{2t^2}dt = \int \frac{1}{t}dt = \log t = \log(x+\sqrt{x^2-1}).$$

To be able to change the variables, we consider the asymptotic line $y = \pm x$. where $y^2 = x^2-1$. We note two lines which are parallel to the asymptotic line: $y = x+t$ and $x+y = t$. The asymptote intersect with $x+y=t$ at $(x,y) = (\frac{t^2+1}{2t}, \frac{t^2-1}{2t})$.

Other method of changing the variables. The line that passes through the point $(-1,0)$ on $y^2 = x^2-1$ can be given by $y = t(x+1)$. The intersecting point between the given line and $y^2 = x^2-1$ other than $(-1,0)$ can be given by $(x,y) = (\frac{1+t^2}{1-t^2}, \frac{2t}{1-t^2})$.