December 17

Algorithm II. Change of variables.

Common parts of the sphere and the cylinder.

$$
\begin{aligned}
& 2\int_{(x-\frac{1}{2})^2+y^2\le\frac{1}{4}} \sqrt{1-x^2-y^2}dxdy = 2\int_{0\le r\le\cos\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}} \sqrt{1-r^2}rdrd\theta \\
= & 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\cos\theta} \sqrt{1-r^2}rdr = 4\int_0^{\frac{\pi}{2}} [-\frac{1}{3}\sqrt{1-r^2}^3]_0^{\cos\theta} d\theta \\
= & \frac{4}{3}\int_0^{\frac{\pi}{2}} (1-\sin^3\theta)d\theta = \frac{2}{3}\pi - \frac{4}{3}\int_0^{\frac{\pi}{2}} (\sin\theta - \cos^2\theta\sin\theta)d\theta \\
= & \frac{2}{3}\pi - \frac{4}{3}[-\cos\theta + \frac{1}{3}\cos^3\theta]_0^{\frac{\pi}{2}} = \frac{2}{3}\pi - \frac{8}{9}.
\end{aligned}
$$

A general case of coordinate transformation.

$x = g(s,t), y = h(s,t)$.

If a treatment $(s,t)\mapsto(x,y)$ provides a one-to-one correspondence between a subset E of st-plane and a subset D of xy-plane, then we can write

$$\int_D f(x,y)dxdy = \int_E f(g(s,t),h(s,t))|J(s,t)|dsdt.$$

Jacobian,

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{pmatrix} \dfrac{\partial g(s,t)}{\partial s} & \dfrac{\partial g(s,t)}{\partial t} \\ \dfrac{\partial h(s,t)}{\partial s} & \dfrac{\partial h(s,t)}{\partial t} \end{pmatrix}.$$

$$J(s,t) = \det\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial g(s,t)}{\partial s}\frac{\partial h(s,t)}{\partial t} - \frac{\partial g(s,t)}{\partial t}\frac{\partial h(s,t)}{\partial s}.$$

Example 1.

$$\frac{\partial(r\cos\theta, r\sin\theta)}{\partial(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \quad J(r,\theta) = r.$$

Example 2. $\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} s \\ t \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can write

$$\frac{\partial(x,y)}{\partial(s,t)} = A.$$

Geometrical meaning of Jacobian: local area dilation ratio.

Area of parallelogram composed of vectors $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} = \left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right|$

$$\int_{\frac{1}{2}x\le y\le 2x, x+y\le\frac{3}{2}\pi} \cos\frac{2x-y}{3}dxdy$$

Let us now change the variables and write $x = 2s + t, y = s + 2t$ such that

$$\begin{aligned} &= \int_{s\geq 0, t\geq 0, s+t\leq \frac{\pi}{2}} \cos s \cdot 3 ds dt = 3\int_0^{\frac{\pi}{2}} dt \int_0^{\frac{\pi}{2}-t} \cos s ds \\ &= 3\int_0^{\frac{\pi}{2}} [\sin s]_0^{\frac{\pi}{2}-t} dt = 3\int_0^{\frac{\pi}{2}} \cos t dt = 3. \end{aligned}$$

Curved surface area ([Takahashi, Kato] p.172, [Kaneko] II p.182, 9.3, [Kodaira] II p.471, § 9.2)

$$S = \int_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1} dx dy.$$