December 10 Algorithm II. Change of variables ([Takahashi, Kato] p.156, [Kaneko] II p.84, 7.3, [Kodaira] II p.371, § 7.3 c))

The single variable case.

Coordinate transformation on a plane. A polar coordinate.

If a treatment $(r, \theta) \mapsto (x, y) = (r\cos\theta, r\sin\theta)$ provides a one-to-one correspondence between a subset E of $r\theta$-plane and a subset D of xy-plane, then we can write

$$\int_D f(x, y)dxdy = \int_E f(r\cos\theta, r\sin\theta)rdrd\theta.$$

The volume of the unit sphere:

$$2\int_{x^2+y^2\leq 1} \sqrt{1-x^2-y^2}dxdy = 2\int_{0\leq r\leq 1, 0\leq\theta\leq 2\pi} \sqrt{1-r^2}rdrd\theta$$

$$= 2\int_0^1 dr \int_0^{2\pi} \sqrt{1-r^2}rd\theta = 4\pi\int_0^1 \sqrt{1-r^2}rdr = 4\pi[-\frac{1}{3}\sqrt{1-r^2}^3]_0^1 = \frac{4\pi}{3}.$$