December 3 Lecture Schedule

Integration of multivariable function

・Definition

・Consecutive integration

・Application (area, volume, center of gravity, and etc.)

・Change of variables formula

Algorithm: I. Consecutive integration ([Takahashi, Kato] p.150 Theorem 6, [Kaneko] II p.76 Theorem 7.4, [Kodaira] II p.323 Theorem 7.3)

$$\int_D f(x,y)dxdy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy. \qquad (1)$$

[Proof] We show the first equal sign. If we let $F(x) = \int_c^d f(x,y)dy$ then the right hand side of the equation can be written as $\int_a^b F(x)dx$.

The partition of $D = \{(x,y)|a \le x \le b, c \le y \le d\}$ can be given by $a = a_0 \le a_1 \le \cdots \le a_n = b, c = c_0 \le c_1 \le \cdots \le c_m = d$ so, we can write $a_{i-1} \le x_i \le a_i$ and thereby, $F(x_i) = \int_c^d f(x_i,y)dy = \sum_{j=1}^m \int_{c_{j-1}}^{c_j} f(x_i,y)dy$.

The mean value theorem gives $c_{j-1} \le y_{ij} \le c_j$ that satisfies $\int_{c_{j-1}}^{c_j} f(x_i,y)dy = f(x_i,y_{ij})(c_j - c_{j-1})$, and from which $F(x_i) = \sum_{j=1}^m f(x_i,y_{ij})(c_j - c_{j-1})$ can be given.

Thus, $\int_a^b F(x)dx$ represents the limit of $\sum_{i=1}^n F(x_i)(a_i - a_{i-1}) = \sum_{i=1}^n \sum_{j=1}^m f(x_i,y_{ij})(a_i - a_{i-1})(c_j - c_{j-1})$ whose partition is subdivided. The limit of the right hand side is given by $\int_D f(x,y)dxdy$.

The right hand sides of the equation (**??**) are also expressed by $\int_a^b dx \int_c^d f(x,y)dy$, and $\int_c^d dy \int_a^b f(x,y)dx$ respectively.

Geometrical meaning: the volume is determined by integrating the area.

If D is defined by $a \le x \le b$,and $c(x) \le y \le d(x)$,

$$\int_{a \le x \le b, c(x) \le y \le d(x)} f(x,y)dxdy = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)dy.$$

Example: volume of the unit sphere

$$2\int_{x^2+y^2\le 1} \sqrt{1-x^2-y^2}dxdy = \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}dy$$
$$= \int_{-1}^1 \pi(1-x^2)dx = \pi[x - \frac{x^3}{3}]_{-1}^1 = \frac{4\pi}{3}.$$

Example: [Takahashi, Kato] p.155 Problem 7 (2)

$$\int_{0\leq y\leq x\leq \pi} y\cos(x-y)dxdy = \int_0^\pi dy \int_y^\pi y\cos(x-y)dx = \int_0^\pi y[\sin(x-y)]_y^\pi dy$$

$$= \int_0^\pi y\sin y dy = [-y\cos y]_0^\pi - \int_0^\pi -\cos y dy = \pi.$$

$$= \int_0^\pi dx \int_0^x y\cos(y-x)dy = \int_0^\pi ([y\sin(y-x)]_0^x - \int_0^x \sin(y-x)dy)dx$$

$$= \int_0^\pi (\int_0^x \sin y dy)dx = \int_0^\pi [-\cos y]_0^x dx = \int_0^\pi 1-\cos x dx = \pi.$$