November 26 Lecture Schedule

Integration of multivariable function and its definition

・Consecutive integration

・Application (area, volume, center of gravity, and etc.)

・Change of variables formula

Definition ([Takahashi, Kato] p.146, 5.1, [Kaneko] II p.72, 7.2, [Kodaira] II p.317, 7.1)

Since it is quite difficult to demonstrate the general cases, we use a rectangular to explain. Let us suppose $D = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$ and let $f(x, y)$ be a continuous function defined on D.

Note that $a = a_0 \leq a_1 \leq \cdots \leq a_n = b, c = c_0 \leq c_1 \leq \cdots \leq c_m = d$ defines a partition of D. We define the fineness of the partition as the maximum value of $a_1 - a_0, \ldots, a_n - a_{n-1}, c_1 - c_0, \ldots, c_m - c_{m-1}$. We take point (x_{ij}, y_{ij}) in each small rectangular $D_{ij} = \{(x, y)|a_{i-1} \leq x \leq a_i, c_{j-1} \leq y \leq c_j\}$.

The sum:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}, y_{ij})(a_i - a_{i-1})(c_j - c_{j-1})$$

, in which a limit for subdivision of the partition is called integral of $f(x, y)$ on D, and we can express that as

$$\int_D f(x, y)dxdy$$

.

We often call this integral a multiple integral just to emphasize that the integral has two variables as we express $\iint_D f(x, y)dxdy$.

Kodaira expresses this integral as $\int_a^b \int_c^d f(x, y)dxdy$ and one must carefully identify which indicates x and y.

Convergence. We let a minimum value be m_{ij} and a maximum value be M_{ij} for $f(x, y)$ on D_{ij} such that we can write as

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}(a_i - a_{i-1})(c_j - c_{j-1}) &\leq \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}, y_{ij})(a_i - a_{i-1})(c_j - c_{j-1}) \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}(a_i - a_{i-1})(c_j - c_{j-1}).\end{aligned}$$

Thus, the convergence follows from

$$(M_{ij} - \text{maximum value of} m_{ij})(b-a)(d-c) \to 0$$

in which subdividing the partition.

In here we used the fact that the maximum value of $(M_{ij} - m_{ij}) to 0$ has the continuous function on a compact set D which is uniformly continuous though, we omit further details.

Linearity, additivity, and positivity.

Algorithm: I. Consecutive integration ([Takahashi, Kato] p.150 Theorem 6, [Kaneko] II p.76 Theorem 7.4, [Kodaira] II p.323 Theorem 7.3)

$$\int_D f(x,y)dxdy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy. \quad (1)$$