November 19 Lecture Schedule

2Extreme values of two variables function (cont.) ([Takahashi, Kato] p.138, 4.5, [Kaneko] II p.11, 6.3, [Kodaira] II n/a)

Theorem. $f(x, y)$ is the second-order continuous and differentiable.

We let $f_x(a, b) = f_y(a, b) = 0$ and $A = f_{xx}(a, b), B = f_{xy}(a, b), C = f_{yy}(a, b)$.

In such a case:

1. $f(x, y)$ may have true local minima at $(x, y) = (a, b)$ if $AC - B^2 > 0$ and $A > 0$.
2. $f(x, y)$ may have true local maxima at $(x, y) = (a, b)$ if $AC - B^2 > 0$ and $A < 0$.
3. $f(x, y)$ may have no extreme values at $(x, y) = (a, b)$ if $AC - B^2 < 0$.

Lemma. Let A, B ,and C be the real numbers, and let $f(x, y) = \frac{1}{2}(Ax^2 + 2Bxy + Cy^2)$. In such a case, we can write $A = f_{xx}(a, b), B = f_{xy}(a, b), C = f_{yy}(a, b)$ so that:

1. $f(x, y)$ may have true minimum values at $(x, y) = (0, 0)$ if $AC - B^2 > 0$ and $A > 0$.
2. $f(x, y)$ may have true maximum values at $(x, y) = (0, 0)$ if $AC - B^2 > 0$ and $A < 0$.
3. $f(x, y)$ may have no extreme values at $(x, y) = (0, 0)$ if $AC - B^2 < 0$.

[Proof] I. Completion of the square: We let

$$Ah^2 + 2Bhk + Ck^2 = \frac{1}{A}((Ah + Bk)^2 + (AC - B^2)k^2) \geq 0$$

.

II. Orthogonalization We let

$${}^tP \begin{pmatrix} A & B \\ B & C \end{pmatrix} P = \begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix}$$

.

[Proof of the theorem] 1. Given the Taylor 's theorem, $0 < t < 1$ exists and satisfies

$$\begin{aligned} & f(a+h,b+k) - f(a,b) \\ = & \frac{1}{2} f_{xx}(a+th,b+tk)h^2 + f_{xy}(a+th,b+tk)hk + \frac{1}{2} f_{yy}(a+th,b+tk)k^2 \end{aligned}$$

.

Note that the function is second-order continuous and differentiable thus, we may assume $f_{xx}(x,y) > 0$ and $f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 > 0$.

If we let

$$A' = f_{xx}(a+th,b+tk), B' = f_{xy}(a+th,b+tk), C' = f_{yy}(a+th,b+tk)$$

, we can write as

$$A'h^2 + 2B'hk + C'k^2 \geq 0$$

, and the equal sign here is identical to $(h,k) = (0,0)$.

Proof of 2. can be done in the same way.

3. If $AC - B^2 < 0$, there exists (h,k) satisfying $Ah^2 + 2Bhk + Ck^2 > 0$ and (h',k') satisfying $Ah'^2 + 2Bh'k' + Ck'^2 > 0$.

If we let $F(t) = f(a+ht, b+kt)$ then we have $F''(0) > 0$. Thus, $F(t)$ holds the local minima at $t = 0$. Now if we let $G(t) = f(a+h't, b+k't)$ then $G(t)$ holds the local maxima at $t = 0$ and thus $f(x,y)$ does not have extreme values at $(x,y) = (a,b)$.