November 12 Lecture Schedule

Change in the order of calculation ([Takahashi, Kato] p.118 Theorem 6, [Kaneko] II p.56 Theorem 6.2, [Kodaira] II p.272 Theorem 6.7)

Theorem. If $f(x, y)$ is the second-order continuous and differentiable then $f_{xy}(a, b) = f_{yx}(ab)$.

[Proof] Let $f(x, y) - f(x, b) = g(x)$, we can write

$$\begin{aligned} & f(x,y) - f(a,y) - f(x,b) + f(a,b) = g(x) - g(a) = g'(s)(x-a) \\ = & (\frac{\partial}{\partial x} f(s,y) - \frac{\partial}{\partial x} f(s,b))(x-a) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(s,t)\right)(x-a)(y-a) \end{aligned}$$

So, if $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right)$ is continuous then we can write as

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,y) - f(x,b) + f(a,b)}{(x-a)(y-a)} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(a,b)\right)$$

Taylor 's Theorem ([Takahashi, Kato] p.132 Theorem 12, [Kaneko] II p.22, [Kodaira] II p.284 § 6.2 f))

There is t, $0 < t < 1$, which satisfies

$$\begin{aligned} & f(a+h, b+k) \\ = & f(a,b) + f_x(a,b)h + f_y(a,b)k + \\ & \frac{1}{2} f_{xx}(a,b)h^2 + f_{xy}(a,b)hk + \frac{1}{2} f_{yy}(a,b)k^2 + \cdots \\ & \sum_{m=0}^{n} \frac{1}{(n-m)!m!} \frac{\partial^n}{\partial x^{n-m} \partial y^m} f(a+ht, b+kt) h^{n-m} k^m \end{aligned}$$

To prove the above, we apply Taylor 's theorem to the single variable function of t, $F(t) = f(a + ht, b + kt)$.

Extreme value of two variables function ([Takahashi, Kato] p.138, 4.5, [Kaneko] II p.11, 6.3, [Kodaira] II n/a)

For $f(x, y)$ to have a local minimum value at $(x, y) = (a, b)$, we can take a circle with a center at (a, b) to be sufficiently small, then we can write $f(a, b) \leq f(x, y)$ for all points (x, y) in the circle thereby, a minimum value exists in the circle. We can explain about the local maximum value in the same way we did for the local minimum. Existence of the extreme values indicates the existence of either local minimum or local maximum values.

Necessary condition: $f(x,y)$ is partial differentiable. If $f(x,y)$ has extreme values at $(x,y) = (a,b)$ then $f_x(a,b) = f_y(a,b) = 0$.

This is obvious if we consider a case of single variable functions $f(x,b)$,and $f(a,y)$.